# Synchronizing Processes on a Tree Network[1]

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[1] Example and several slides from J. R. Abrial book *Modeling in Event-B: system and software engineering*.

# Purpose of this lecture

- We will formalize the solution to a problem in distributed computing.
  - Studied in: *W.H.J. Feijen and A.J.M. van Gasteren. On a Method of Multi-programming. Springer Verlag, 1999.*
- Using and updating functions.
- Formalize and prove properties on an interesting structure: a tree.
- Proofs more complex than those seen so far.

As usual:

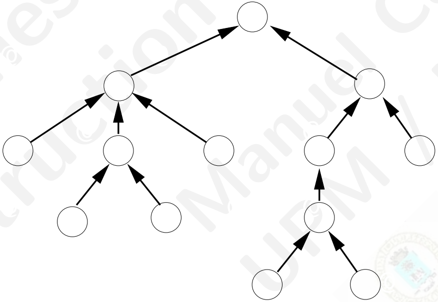
- Define the informal requirements
- Define the refinement strategy
- Construct the various more and more concrete models

## Comparison with previous examples

- Not a transformational system.
  - Not supposed to finish.
  - No final result.
- Not reactive.
  - No *external* world that reacts to system changes.
- Distributed.
  - Different *nodes* act autonomously.
  - With limited information access.
  - However, communication assumed to be reliable.
- Internal concurrency.
  - Every node has concurrent processes.
- Small model: just three events in the last refinement.
- However, proofs and reasoning are comparatively complex.

# Requirements

| ENV 1 | We have a fixed set of processes forming a tree |
|---|---|



- Note: they do not need to form a tree from the beginning.
- A set of communicating processes can coordinate to form a tree.

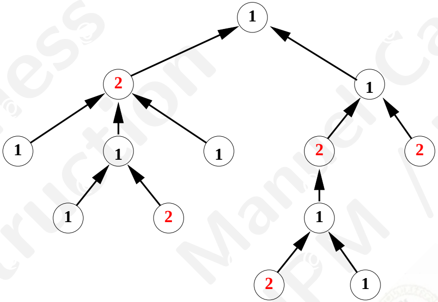
# Requirements (Cont.)

- All processes are supposed to execute forever the same code.
- But processes must remain (somewhat) synchronized.
- For this, each process has (initially) one counter.

| | |
|---|---|
| ENV 2 | Each process has a counter, which is a natural number |

- A process counter represents its "phase"
  (related to the work for which they have to synchronize).
- Difference between any two counters $\leq$ one.
  - Each process is thus at most one phase ahead of the others

# Requirements (Cont.)




| FUN 3 | The difference between any two counters is at most one |
| --- | --- |

# Requirements (Cont.)

- Reading the counters

| | |
|---|---|
| FUN 4 | Each process can read the counters of its neighbors only |

(*Neighbors* to be understood as *connected by a link*)

- Modifying the counters

| | |
|---|---|
| FUN 5 | The counter of a process can be modified by this process only |

# Refinement strategy

- Construct abstract initial model dealing with FUN 3 and FUN 5
- Improve design to (partially) take care of FUN 4
- Improve design to better take care of FUN 4
- (Simplify final design to obtain efficient implementation).

| | |
|---|---|
| FUN 3 | The difference between any two counters is at most one |

| | |
|---|---|
| FUN 4 | Processes read counters of immediate neighbors only |

| | |
|---|---|
| FUN 5 | A process can modify only its counter(s) |

# Steps

1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

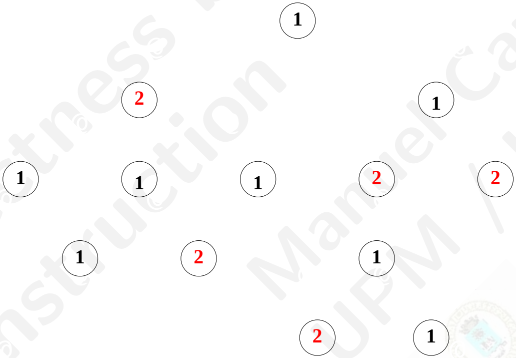
# Initial model: the state

- Simplify situation: forget about tree
- We just define the counters and express the main property: FUN 3

| | |
|---|---|
| FUN 3 | The difference between any two counters is at most one |

- The initial model is always far more abstract than the final system
- Other requirements are probably not fulfilled

# Abstract situation




| | |
|---|---|
| FUN 3 | The difference between any two counters is at most 1 |

Suggest constants, axioms, variables, invariants for an initial model!

# Initial model: the state

**carrier set:** $P$

**axm0_1:** $\text{finite}(P)$

**variable:** $c$

**inv0_1:** $c \in P \rightarrow \mathbb{N}$

**inv0_2:** $\forall x, y \cdot \left( \begin{array}{l} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{array} \right)$

- ✓ *Create project* `synch_tree`
- ✓ *Create context* `c0` *with set, axiom*
- ✓ *Create machine* `m0` *with variable, invariants.*

## Is that right?

- inv0_2 may be surprising:

$$\mathcal{I}_0 : \forall x, y \cdot x \in P \wedge y \in P \Rightarrow c(x) \leq c(y) + 1$$

- Is it the same as $\mathcal{I}_1 : \forall i, j \cdot |c(i) - c(j)| \leq 1$?

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**Proof by double implication.**

Let us choose two arbitrary nodes with counters $a$ and $b$.

- If the invariant holds, then $a \leq b + 1$ and $b \leq a + 1$. From there, $a - b \leq 1$ and $b - a \leq 1$, therefore $|a - b| \leq 1$, and $\mathcal{I}_0 \Rightarrow \mathcal{I}_1$.
- If $|a - b| \leq 1$, then both $a - b \leq 1$ and $b - a \leq 1$. Then, inv0_2 is implied by the intended invariant, and $\mathcal{I}_1 \Rightarrow \mathcal{I}_0$.

# Initial model: events

init
$$c := P \times \{0\}$$

ascending
**any** $n$ **where**
$n \in P$
$\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
**then**
$c(n) := c(n) + 1$
**end**

- Note: any $n$ is logically $\forall n \cdot n \in P \wedge \cdots$
  - $\forall$ can appear in guards.
  - any introduces $\forall$ whose scope is the whole event.
- Intuition: *Any increment that respects difference among nodes can be done.*
- Does not mean all increments are executed: non-determinism!
- Not final state (there is none): action that (hopefully) respects invariant.

✓ *Add initialization, event*

Note: $\times$ is entered with $**$, any with pull-down menu, “Add event parameter”.

# Proof of invariant preservation

$c \in P \rightarrow \mathbb{N}$ **inv0_1**

$\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{pmatrix}$ **inv0_2**

$n \in P$

$\forall m \cdot (m \in P \Rightarrow c(n) \leq c(m))$

Guards of event ascending

$\vdash$

$\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ (c \mathbin{\lhd\!\!-} \{n \mapsto c(n) + 1\})(x) \leq (c \mathbin{\lhd\!\!-} \{n \mapsto c(n) + 1\})(y) + 1 \end{pmatrix}$

$\Uparrow$

Modified invariant **inv0_2**

In Rodin: automatic; if not, repeatedly apply lassoing, p0 or m0.

# Model so far

**CONTEXT** c0
**SETS**
  P
**AXIOMS**
  axm1: $finite(P)$
**END**

**MACHINE** m0
**SEES** c0
**VARIABLES**
  c
**INVARIANTS**
  inv1: $c \in P \to \mathbb{N}$
  inv2: $\forall x, y \cdot x \in P \land y \in P \Rightarrow c(x) \leq 1 + c(y)$
**EVENTS**
**Initialisation**
  **begin**
    act1: $c := P \times \{0\}$
  **end**
**Event** ascending ⟨ordinary⟩ $\widehat{=}$
  **any**
    n
  **where**
    grd12: $n \in P$
    grd11: $\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
  **then**
    act11: $c(n) := c(n) + 1$
  **end**

# Problem with the current event

ascending
  **any** $n$ **where**
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
  **then**
    $c(n) := c(n) + 1$
  **end**

What requirement is this event breaking?

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What requirement is this event breaking?

| | |
|---|---|
| FUN 2 | Each node can read the counters of its immediate neighbors only |

## Steps

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4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

# First refinement: (partially) solving the problem

- Introduce a designated process $r$.
- Assume that counter of $r$ always minimal

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

- Rationale:
  - We only synchronize with $r$ — not compliant, but communication restricted.
  - Helps ensure that difference between any two nodes $\leq$ one.
    - If inv0_1 : $\forall x, y \cdot x \in P \land y \in P \Rightarrow c(x) \leq c(y) + 1$ , then $|c(r) - c(m)| \leq 1$ for any $m$.
    - If $c(r) \leq c(m)$, then $c(m) = c(r)$ or $c(m) = c(r) + 1$ for any $m$.
    - Then $|c(m) - c(n) \leq 1|$, for any $m, n$ (will be proved).
- Treat this property as a new (temporary) invariant.

✓ *Extend* `c0` *into* `c1` *(left pane, right click, "Extend"), add constant $r$, axiom $r \in P$*
✓ *Refine* `m0` *into* `m1` *(left pane, right click, "Refine"), add new invariant*
✓ `m0` *should "see"* `c1`

# First refinement: proposal for the event refinement

We simplify the guard

| (abstract-)ascending | (concrete-)ascending |
| --- | --- |
| **any** $n$ **where**<br>$n \in P$<br>$\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$<br>**then**<br>$c(n) := c(n) + 1$<br>**end** | **any** $n$ **where**<br>$n \in P$<br>$c(n) = c(r)$<br>**then**<br>$c(n) := c(n) + 1$<br>**end** |

- Note: if $c(r)$ minimal, $c(n) < c(r)$ impossible; therefore $c(n) = c(r)$

  ✓ *Change "extended" to "not extended", change guard*

- We have then to prove guard strengthening.

# Guard strengthening

| | |
|---|---|
| $c \in P \rightarrow \mathbb{N}$ | **inv0_1** |
| $\forall x, y \cdot \left( \begin{array}{l} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{array} \right)$ | **inv0_2** |
| $\forall m \cdot (\, m \in P \;\Rightarrow\; \boxed{c(r)} \leq c(m)\,)$ | **new invariant** |
| $n \in P$<br>$\boxed{c(n) = c(r)}$ | Guards of concrete event ascending |
| $\vdash$ | |
| $n \in P$<br>$\forall m \cdot (\, m \in P \Rightarrow \boxed{c(n)} \leq c(m)\,)$ | Guards of abstract event ascending |

In Rodin: lasso + p0

✓ *Go to the proving perspective, discharge proof*

# Model so far

`inv1` not discharged.

**CONTEXT** c1
**EXTENDS** c0
**CONSTANTS**
  r
**AXIOMS**
  axm1: $r \in P$
**END**

**MACHINE** m1
**REFINES** m0
**SEES** c1
**VARIABLES**
  c
**INVARIANTS**
  inv1: $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$
**EVENTS**
**Initialisation** ⟨extended⟩
  **begin**
    act1: $c := P \times \{0\}$
  **end**
**Event** ascending ⟨ordinary⟩ $\widehat{=}$
**refines** ascending
  **any**
    n
  **where**
    grd1: $n \in P$
    grd2: $c(r) = c(n)$
  **then**
    act1: $c(n) := c(n) + 1$
  **end**
**END**

# Pending problems

```
ascending
  any n where
    n ∈ P
    c(n) = c(r)
  then
    c(n) := c(n) + 1
  end
```

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

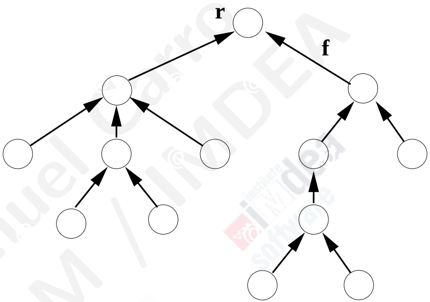
1. Prove that new “invariant” is preserved by the event
2. The guard of the event still does not fulfill requirement FUN 4.

| | |
|---|---|
| FUN 4 | Each node can read the counters of its immediate neighbors only |

- Problem 1 solved in this refinement
- Problem 2 solved later

# First refinement: defining the tree

- Tree: root $r$ and "pointer" $f$ from each node in $P \setminus \{r\}$ to every node's parent.
- Plus some additional properties and inference rules.
- Reminder: sets, relations, functions, specific data structures and their inference rules.
- Note: constructing a tree ($\equiv$ root / leader + links) is a classical problem in distributed systems.
- Can also be tackled using Event B.




Invariant: we have a condition involving nodes in pairs and we need a condition that relates a single node $r$ with all the others.

How can we model the relation node / parent node?

# Update model

✓ *Add to c1 (note $f$ is $\twoheadrightarrow$, written -->>)*

- Constant $f$.
- Axioms:

$$L \subseteq P$$
$$f \in P \setminus \{r\} \twoheadrightarrow P \setminus L$$
$$\forall S \cdot S \subseteq f^{-1}[S] \Rightarrow S = \varnothing$$

- $f^{-1}$ is written f~.
- $\twoheadrightarrow$: $f$ defined for all $P \setminus \{r\}$ and *arrives* to every element in $P \setminus L$.

# Minimal counter at the root

- Minimality of counter at the root

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

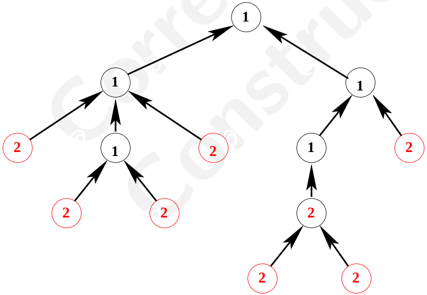
  relates $c(r)$ with $c(m)$ for every $m$.
- Events change local values and consult neighbouring values.
- We can (easily) prove properties relating neighbouring nodes.
- How can we relate local properties with global properties?

# Minimal counter at the root

- We define a weaker, local invariant first.
- The counter at every node is not greater than its descendants:

$$\text{inv1\_1} : \forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$$

✓ *Add it to* `m1`




**Rationale (advancing the algorithm)**

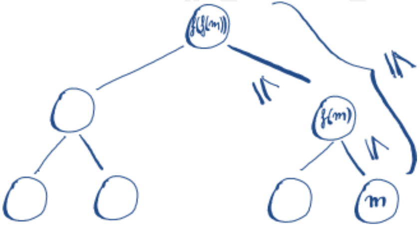
- Assume we can update the tree keeping a maximum difference between neighbors.
- Will be helpful to prove $c(r) \leq c(m)$.

# Minimal counter at the root: induction

We need to *extend* the local property

$$\forall m \cdot m \in P \backslash \{r\} \Rightarrow c(f(m)) \leq c(m)$$

to the whole tree.




- Start with leaves $l \in L$.
- Prove that for any $l$, $c(f(l)) \leq c(l)$, then $c(f(f(l))) \leq c(f(l)) \leq c(l), \ldots$
- Work upwards towards root $r$.

**OR**

- Start with $r$.
- Prove that for all $m$ s.t. $r = f(m)$, $c(r) \leq c(m)$.
  $m$ is a child of $r$
- Then, for all $m'$ s.t. $m = f(m')$, $c(m) \leq c(m') \ldots$
- And so on towards the leaves.

# Minimal counter at the root: induction

- Induction: difficult for theorem provers to do on their own.
  - Needs to identify base case, property to use for induction.
- Then, proving property given base case & inductive step within theorem provers' capabilities.
- In Rodin: needs adding induction scheme:

  ✓ *Add to `c1`:*
  $\forall S \cdot S \subseteq P \land r \in S \land (\forall n \cdot n \in P \setminus \{r\} \land f(n) \in S \Rightarrow n \in S) \Rightarrow P \subseteq S$

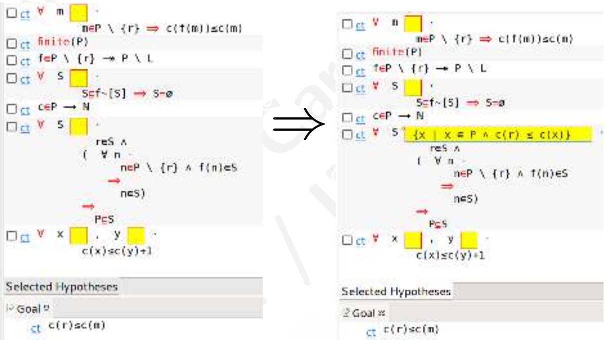
  ✓ *Tip: `Ctrl-Enter` breaks text in input box in separate lines.*
- Instantiating it with the property to prove, expressed as a set:
  $\{x \mid x \in P \land c(r) \leq c(x)\}$ (next slide)

✓ *In `m1`: ensure you have* inv1_1: $\forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$

✓ *Ensure* thm1_1: $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$ *below invariant, marked as theorem*

# Induction in Rodin: instantiation

- Double click in the unproved theorem (left pane).
- Switch to prover view, lasso.
- Locate induction axiom.
- Enter
  `{x | x ∈ P ∧ c(r) ≤ c(x)}`.
- Return and p0.
- The theorem should be proved now.



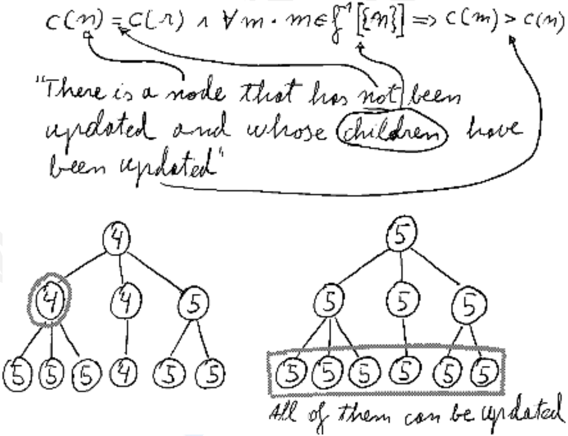

Invariant inv1_1 not yet proved. Requires order between parent and children $c(f(m)) \leq c(m)$ that ascending cannot guarantee: guard $c(r) = c(n)$ allows updates in arbitrary order. Will enforce through more local comparison.

# More local comparison

- Nodes with difference $\leq$ one from $r$.
- When can we update looking locally?

ascending
  any $n$ where
    $n \in P$
    $c(r) = c(n)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  then
    c(n) := c(n) + 1
  end

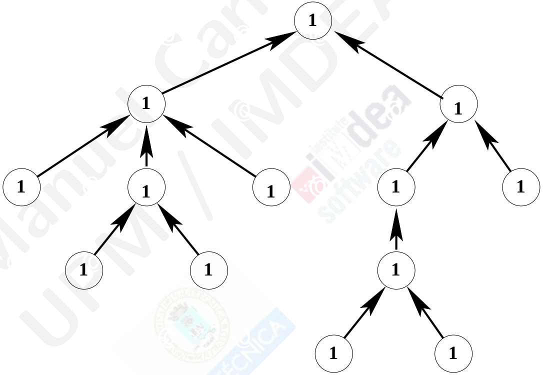
Ensure inv1_1 is preserved: double click, prover view, lasso, p0 should do it.



# How it is expected to work

Update order restricted:

- **Before:** any node whose counter is equal to the root (the one with the minimum).
- **Now:** only those nodes whose counters are, in addition, smaller than all its descendants.
- Updates will go in waves towards the root.



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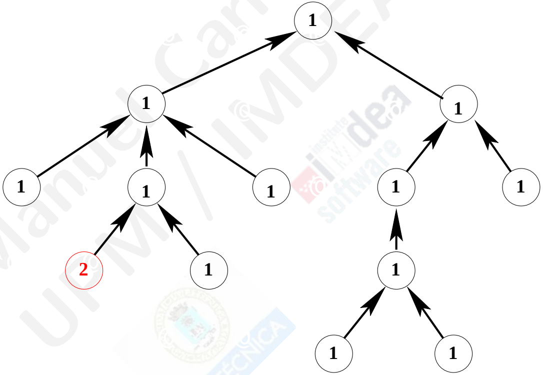

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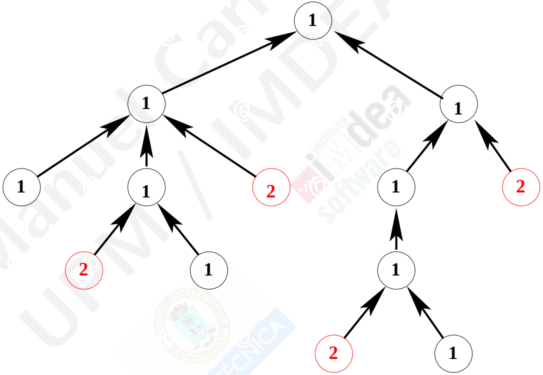

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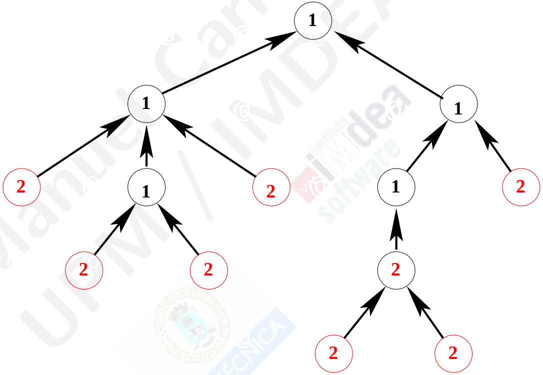

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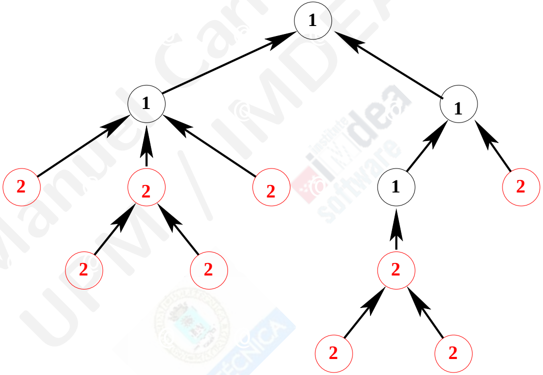

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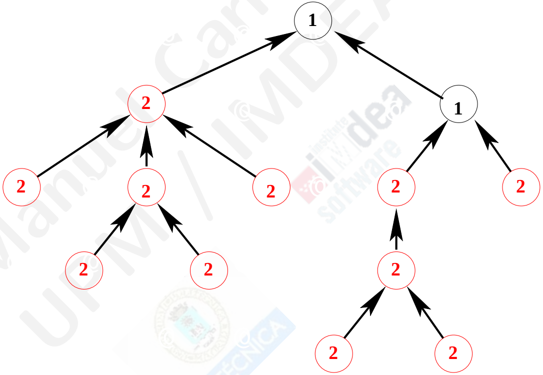

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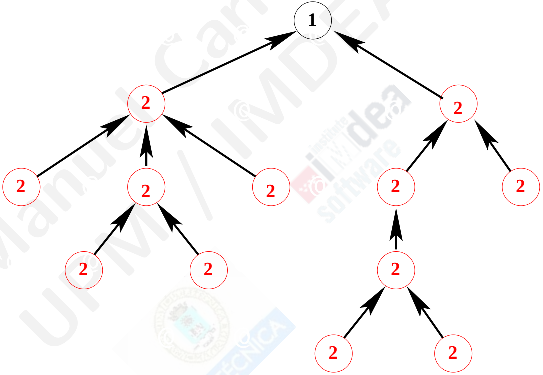

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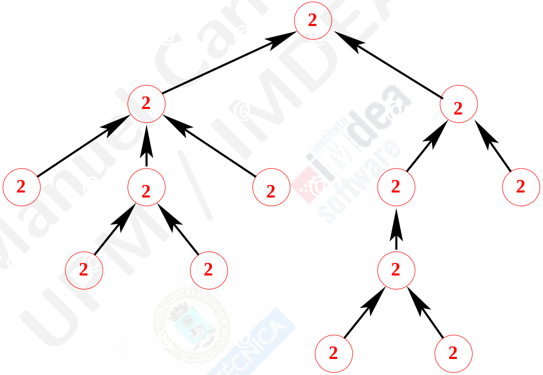

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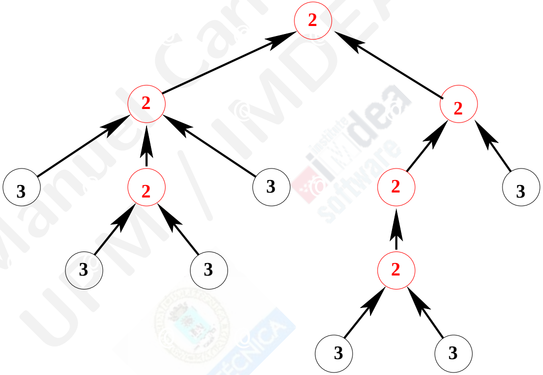

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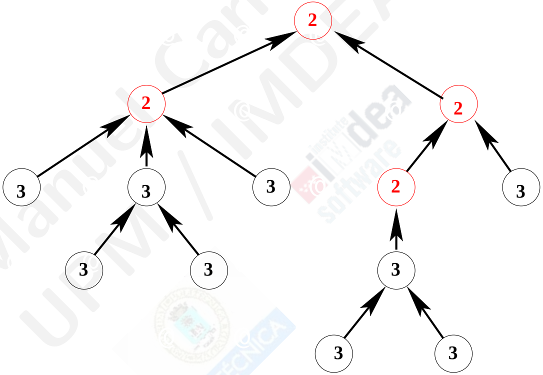

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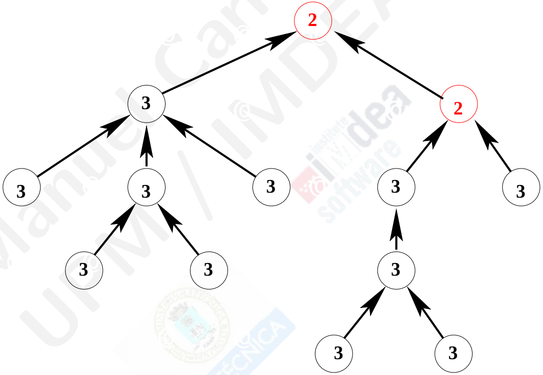

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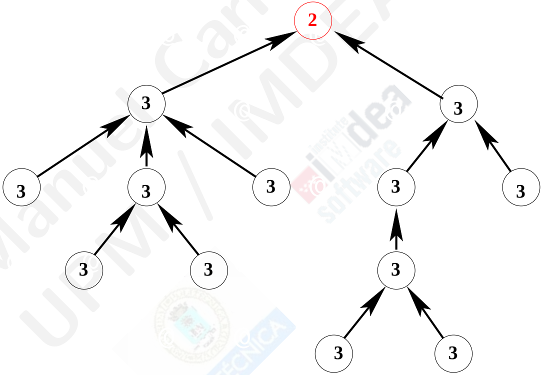
- **Now:** only those nodes whose counters are, in addition, smaller than all its descendants.
- Updates will go in waves towards the root.



# Neighborhood checking

| | |
|---|---|
| FUN 4 | Each process can read the counters of its immediate neighbors only |

- $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$ uses only local comparisons.
- $c(r) = c(n)$ uses non-local comparisons.
- We will tackle that in the next refinement.

# Model so far

**CONTEXT** c1
**EXTENDS** c0
**CONSTANTS**

r

f

L

**AXIOMS**

axm1: $r \in P$

axm3: $L \subseteq P$

Leaves

axm2: $f \in P \setminus \{r\} \twoheadrightarrow P \setminus L$

axm4: $\forall S \cdot S \subseteq f^{-1}[S] \Rightarrow S = \varnothing$

axm5:

$\forall S \cdot S \subseteq P \land$
$r \in S \land$
$(\forall n \cdot n \in P \setminus \{r\} \land f(n) \in S \Rightarrow n \in S)$
$\Rightarrow$
$P \subseteq S$

**END**

**MACHINE** m1
**REFINES** m0
**SEES** c1
**VARIABLES**

c

**INVARIANTS**

inv1: $\forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$

inv2: ⟨theorem⟩ $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$

**EVENTS**

**Initialisation** ⟨extended⟩

**begin**

act1: $c := P \times \{0\}$

**end**

**Event** ascending ⟨ordinary⟩ $\widehat{=}$
**refines** ascending

**any**

n

**where**

grd1: $n \in P$

grd2: $c(r) = c(n)$

grd3: $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$

**then**

act1: $c(n) := c(n) + 1$

**end**

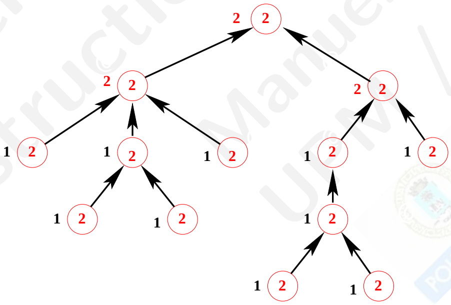
**END**

## Steps

1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

## Second refinement

- Replace the guard $c(r) = c(n)$.
- Processes must be aware when this situation does occur.
- Add second counter $d(\cdot)$ to each node to capture value of $c(r)$.



# Second refinement: the state

carrier set: $P$

constants: $r, f$

variables: $c, d$

Invariant **inv2_2** is as **inv0_2**

**inv2_1:** $d \in P \to \mathbb{N}$

**inv2_2:** $\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ d(x) \leq d(y) + 1 \end{pmatrix}$

$d$ has an overall property similar to $c$:

$\forall x, y \cdot x \in P \land y \in P \Rightarrow c(x) \leq c(y) + 1$

- $d$ will capture the value of $c(r)$.
- It will be updated in a downward wave.

✓ *Refine m1 into m2*

✓ *Add variable d and invariants*

# Updating $d$

This refinement captures:

- The existence of $d$.
- How its update can proceed not to break its invariant.

Event descending
  any $n$ where
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow d(n) \leq d(m)$
  then
    $d(n) := d(n) + 1$
  end

✓ *Add event to* `m2`

✓ *Initialize* $d$ *to 0 (copy the initialization of* $c$*)*

# Steps

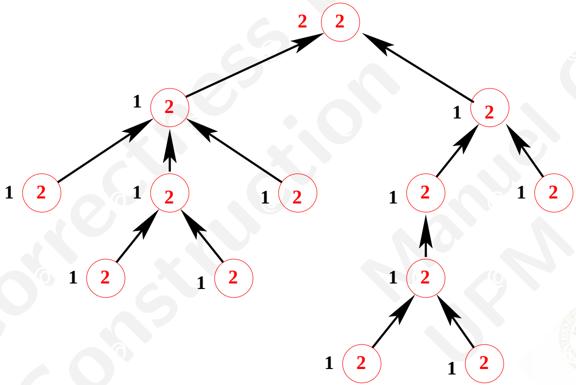
1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

# Third refinement

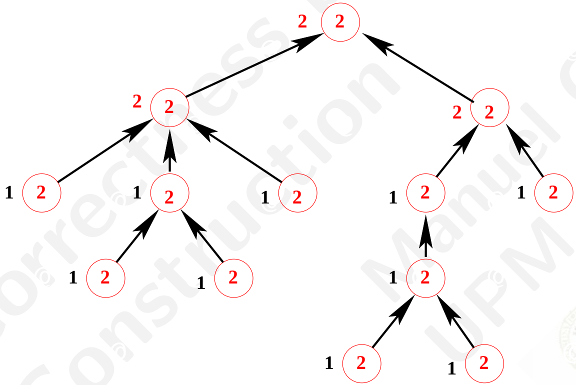
- We extend the invariant of counter $d$.
- We establish the relationship between both counters $c$ and $d$.
  - This will allow us to refine event ascending
- We construct the descending wave (by refining event descending).
- Remark: this is the most difficult refinement.

✓ *Refine* `m2` *into* `m3`

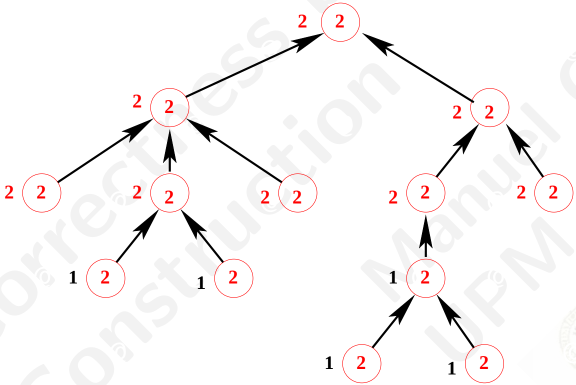
# Idea behind third refinement



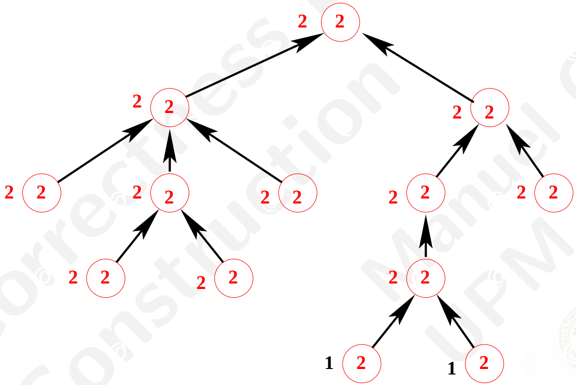
# Idea behind third refinement



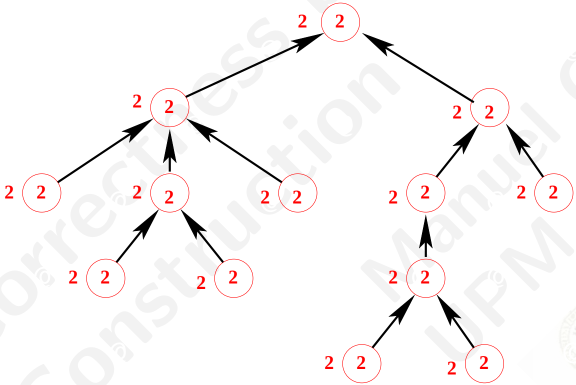
# Idea behind third refinement



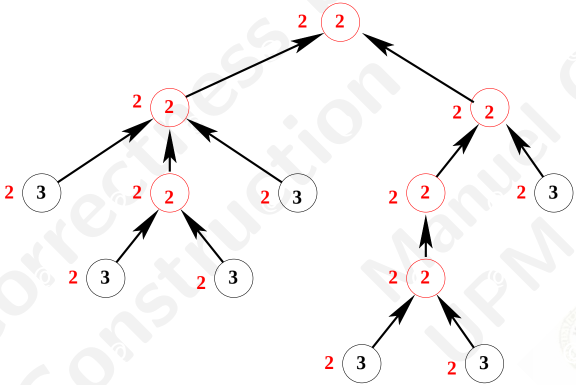
# Idea behind third refinement



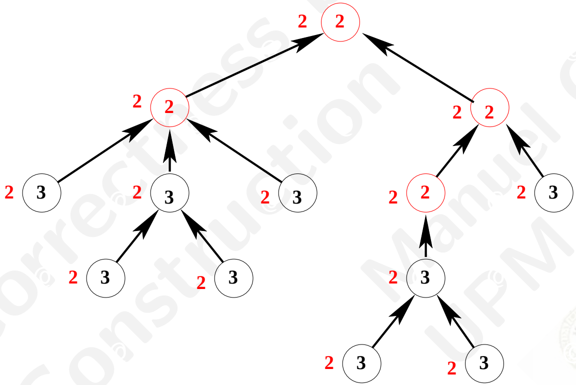
# Idea behind third refinement



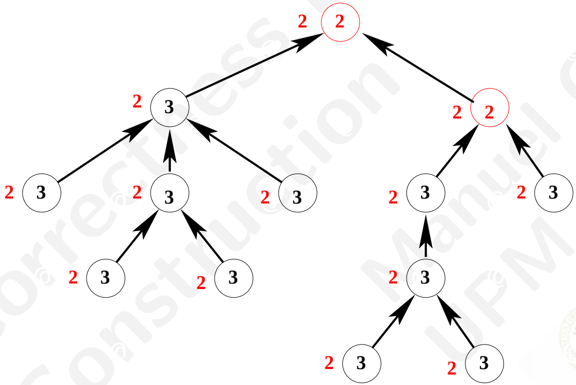
# Idea behind third refinement



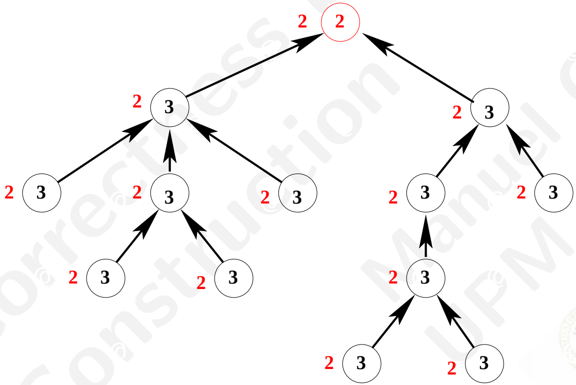
# Idea behind third refinement



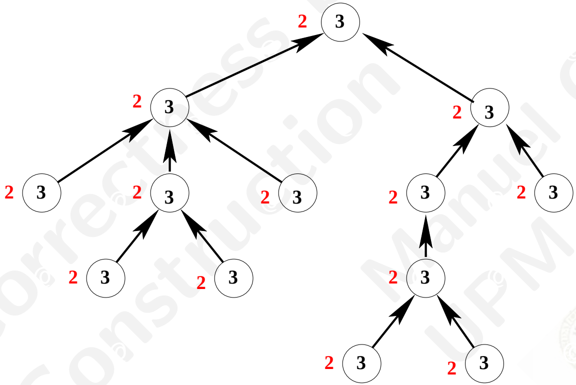
# Idea behind third refinement



# Idea behind third refinement



# Idea behind third refinement



# State and invariants

- Recall local condition for $c$:

$$\text{inv1\_1} : \forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$$

*Every node's counter is smaller than or equal to its children's.*

- Local condition for $d$ is similar:

$$\text{inv3\_1} : \forall m \cdot m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m))$$

*Every node's counter is smaller than or equal to its parent (if it has a parent).* This is what makes the wave *descending*.

- inv3_1 and tree induction proves that the root has the highest value of $d(\cdot)$:

$$\text{thm3\_1} : \forall n \cdot n \in P \Rightarrow d(n) \leq d(r)$$

*(remember: root had the smallest value of $c(\cdot)$)*

# Proving theorem and invariant

- ✓ *Add to* `m3`*:*

$$inv3\_1: \quad \forall m \cdot m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m))$$
$$thm3\_1: \quad \forall n \cdot n \in P \Rightarrow d(n) \leq d(r)$$

- ✓ *Mark the latter as theorem*
- ✓ *Double click on the PO for THM*
- ✓ *Go to proving perspective; locate induction axiom*
- ✓ *Instantiate with* $\{x | x \in P \wedge d(x) \leq d(r)\}$*, invoke p0*
- ✓ *That should prove* $thm3\_1$
- ✓ $inv3\_1$ *cannot be proved yet - reasons similar to* $c$*.*

*We will deal with that later*

# Refining *ascending*

**Event** (abstract−)ascending
  **any** $n$ **where**
    $n \in P$
    $c(n) = c(r)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  **then**
    $c(n) := c(n) + 1$
  **end**

**Event** (concrete−)ascending
  **any** $n$ **where**
    $n \in P$
    $c(n) = d(n)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  **then**
    $c(n) := c(n) + 1$
  **end**

- Downward wave $d$ will eventually propagate $d(r)$.
  - ✓ *Change event guard in* `m3`

ascending: only local comparisons now!

# Refining *ascending*

Event (abstract−)ascending
any $n$ where
$n \in P$
$c(n) = c(r)$
$\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
then
$c(n) := c(n) + 1$
end

Event (concrete−)ascending
any $n$ where
$n \in P$
$c(n) = d(n)$
$\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
then
$c(n) := c(n) + 1$
end

- Downward wave $d$ will eventually propagate $d(r)$.
  ✓ *Change event guard in* `m3`
- Need to prove guard strengthening.

ascending: only local comparisons now!

# Refining *ascending*

Event (abstract−)ascending
  any $n$ where
    $n \in P$
    $c(n) = c(r)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  then
    $c(n) := c(n) + 1$
  end

- Downward wave $d$ will eventually propagate $d(r)$.
  ✓ *Change event guard in* m3
- Need to prove guard strengthening.
- We cannot. $c$ and $d$ unrelated so far!
  ✓ *Relate* $c$ *and* $d$*:* inv3_2 : $d(r) \leq c(r)$
- If needed: proving perspective, lasso + p0 proves strengthening.

Event (concrete−)ascending
  any $n$ where
    $n \in P$
    $c(n) = d(n)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  then
    $c(n) := c(n) + 1$
  end

ascending: only local comparisons now!

# Refining *descending*

- A different case.
- Two situations raise a change of $d$:
  1. For a non-root node: parent's $d$ change.
  2. For the root node: $c(r)$ changes.
- Different guards.
- We will prepare the events to be edited.

✓ *Change (concrete) descending event to non-extended*

✓ *Left click on circle to left of name to select*
`Ctrl-C` *to copy,* `Ctrl-V` *to paste*

✓ *Rename first event as descending_nr.*

✓ *Rename second event as descending_r.*

# Refining *descending*: the non-root case

Event (abstract−)descending
  any $n$ where
    $n \in P$
    $\forall m \cdot m \in N \Rightarrow d(n) \leq d(m)$
  then
    $d(n) := d(n) + 1$
  end

Event (concrete−)descending
  any $n$ where
    $n \in P \setminus \{r\}$
    $d(n) \neq d(f(n))$
  then
    $d(n) := d(n) + 1$
  end

✓ *Update guards*

(Note: Rodin $\geq$ 3.6 seems to prove strengthening automatically; previous versions needed additional steps [in next slide])

# Proving guard strengthening

Note: the steps below do not seem to be necessary in Rodin 3.6 with the Atelier B provers installed. Strengthening is proven automatically.

$$n \in P \setminus \{r\}, d(n) = d(f(n)), m \in P \vdash d(n) \leq d(m)$$

We need some magic mushrooms to help the provers:

$$\begin{aligned} \text{thm3\_2}: \quad & \forall n \cdot n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1 \\ \text{thm3\_3}: \quad & \forall n \cdot n \in P \Rightarrow d(r) \in d(n)..d(n)+1 \end{aligned}$$

thm3_2 downward wave, parent is at most one more than children (when it has just been increased)

thm3_3 special case for root (the first one to be increased)

# Refining *descending* (Cont. — the root case.)

Event (abstract−)descending
  any $n$ where
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow d(n) \leq d(m)$
  then
    $d(n) := d(n) + 1$
  end

Event (concrete−)descending
  refines
    *descending*
  when
    $d(r) \neq c(r)$
  with
    n: $n = r$
  then
    $d(r) := d(r) + 1$
  end

---

✓ *Click on circle left of param. $n$, delete*

- Parameter $n$ disappeared!
- Substitute *(witness)* for GRD, SIM.
- We are particularizing for $r$.

- Similar to gluing invariant!
- Note with label: specific Rodin idiom.
- Need to prove
  $d(r) \neq c(r), m \in P \vdash d(r) \leq d(m)$

# Finishing proofs

**The technique in this slide was necessary for Rodin versions previous to 3.6. For Rodin 3.6 onwards, it seems that it is not necessary. Skip to the next slide.**

I needed two more magic pills:

$$\text{inv3\_3}: \quad \forall n \cdot n \in P \Rightarrow c(n) \in d(n)..d(n)+1 \quad \text{To prove GRD}$$

$$\text{thm3\_4}: \quad \forall n \cdot n \in P \Rightarrow c(r) \in d(n)..d(n)+1 \quad \text{To prove inv3\_3}$$

Plus, if not added before:

$$\text{thm3\_2}: \quad \forall n \cdot n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1$$

$$\text{thm3\_3}: \quad \forall n \cdot n \in P \Rightarrow d(r) \in d(n)..d(n)+1$$

After this, the invariant can be proved with a combination of several steps:

- Apply lasso.
- Instantiate $\forall n \cdot c(r) \in d(n)..d(n)+1$ (which relates $c$ and $d$) with $n$.
- Remove $\in$ in goal ($c(n) \in d(n)+1..d(n)+1+1$) to create inequalities.
- Do P0 in $c(n) \leq d(n)+1+1$ goal.
- Note that only possibility to prove is $d(n) = c(n)$.
- Do case distinction with $d(n) = c(n)$,
- Apply ML to the subgoals.

## Finishing proofs

**This strategy is necessary with Rodin 3.6 and 3.7 and, apparently, 3.9.**

An additional invariant is necessary to prove GRD of descending_r:

$$\text{inv3\_3}: \quad \forall n \cdot n \in P \Rightarrow c(n) \in d(n)..d(n)+1$$

After adding it, GRD is immediately proven. However, the invariant remains unproven. It can be proved with the following steps:

- Apply lasso.
- Remove $\in$ in goal $c(n) \in d(n)+1..d(n)+1+1$ to transform it into inequalities that can be proven separately.
- Use ml or p0 for the goal $c(n) \leq d(n)+1+1$.
- For $d(n)+1 \leq c(n)$, do case distinction:
  - Either with $d(n) = c(n)$, or
  - with $d(n)+1 = c(n)$

  and ML to the subgoals.

# Third refinement: invariants

**inv3_1:** $\forall m \cdot ( m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m)) )$

**inv3_2:** $d(r) \leq c(r)$

**inv3_3:** $\forall n \cdot ( n \in P \Rightarrow c(n) \in d(n) \,..\, d(n)+1 )$

**thm3_1:** $\forall m \cdot ( m \in P \Rightarrow d(m) \leq d(r) )$

**thm3_2:** $\forall n \cdot ( n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n) \,..\, d(n)+1 )$

**thm3_3:** $\forall n \cdot ( n \in P \Rightarrow d(r) \in d(n) \,..\, d(n)+1 )$

**thm3_4:** $\forall n \cdot ( n \in P \Rightarrow c(r) \in d(n) \,..\, d(n)+1 )$

# Third refinement: events

Event descending_r
  when
    $d(r) \neq c(r)$
  with
    n: $n = r$
  then
    $d(r) := d(r) + 1$
  end

Event descending_nr
  any $n$ where
    $n \in P \setminus \{r\}$
    $d(n) \neq d(f(n))$
  then
    $d(n) := d(n) + 1$
  end

Event ascending
  any $n$ where
    $n \in P$
    $c(n) = d(n)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  then
    $c(n) := c(n) + 1$
  end

# Steps

1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

# Observation

- The difference among counters is at most one.
  - That has been proven by construction.
- In the guards, we only care whether they are equal or not.
- For this, we only need parity!

$$a, b \in \mathbb{N} \wedge |a - b| \leq 1 \Rightarrow (a = b \Leftrightarrow parity(a) = parity(b))$$

- We will prove that this is a valid refinement.

✓ *Extend context c1 into c2*

✓ *Refine m3 into m4*

✓ *m4 should see c2*

# Formalizing parity

- We replace the counters by their parities
- we add the constant $\color{red}{parity}$

**carrier set:** $P$

**constants:** $r, f, parity$

**axm4_1:** $parity \in \mathbb{N} \rightarrow \{0,1\}$

**axm4_2:** $parity(0) = 0$

**axm4_2:** $\forall x \,.\, (\, x \in \mathbb{N} \;\Rightarrow\; parity(x+1) = 1 - parity(x)\,)$

✓ *Add parity and axioms to c2. Note: parity is a function!*

✓ *Need some clicking (dom to $\mathbb{N}$ + ML ) to prove WD*

# The definitions that replace $c(\cdot)$ and $d(\cdot)$

- We replace $c$ and $d$ by $p$ and $q$

**variables:** $p, q$

**inv4_1:** $p \in P \rightarrow \{0, 1\}$

**inv4_2:** $q \in P \rightarrow \{0, 1\}$

**inv4_3:** $\forall n \,.\, (\, n \in P \;\Rightarrow\; p(n) = parity(c(n))\,)$

**inv4_4:** $\forall n \,.\, (\, n \in P \;\Rightarrow\; q(n) = parity(d(n))\,)$

✓ *Do it in* `m4`*. Note the gluing invariants!* `p` *and* `q` *really syntactic sugar.*
✓ *Remove variables* `c` *and* `d`*. Not accessed / updated in this refinement!*
✓ *Initialize* `p` *and* `q`*, remove initializations for* `c` *and* `d`*.*

# New events: counters replaced by parity

ascending
**any** $n$ **where**
$n \in P$
$p(n) = q(n)$
$\forall m \cdot (\ m \in f^{-1}[\{n\}] \ \Rightarrow \ p(m) \neq p(n)\ )$
**then**
$p(n) := 1 - p(n)$
**end**

descending_1
**any** $n$ **where**
$n \in P \setminus \{r\}$
$q(n) \neq q(f(n))$
**then**
$q(n) := 1 - q(n)$
**end**

descending_2
**when**
$p(r) \neq q(r)$
**then**
$q(r) := 1 - q(r)$
**end**

### GRD of $q(n) = p(n)$

- The essence of the pending GRD proof is $\ldots, q(n) = p(n) \vdash c(n) = d(n)$.
- Depends on proving $parity(a) = parity(b) \Rightarrow a = b$.
- Holds in specific cases (if $|a - b| \leq 1$).
- But theorem provers unable to apply / deduce that property.
- Needs to be stated explicitly:

$$\forall x, y \cdot y \in \mathbb{N} \land x \in y..y+1 \quad \Rightarrow$$
$$(parity(x) = parity(y) \Leftrightarrow x = y)$$

- We could make it axiom, but it can be proven as theorem (better!).

- We need to deal with Well Definedness and the theorem itself.

  WD: Removing dom in goal + P0 takes care of it (if WD is to be discharged).
  THM: Adding hypothesis + case distinction works. See below.

- $\Longleftrightarrow$: split in two implications. One is proven.
- For the other: introduce ah with possible values of $x$: $x = y \lor x = y + 1$ (because $x \in y..y + 1$ among the hypotheses).
- Prove new hypothesis with ml.
- For the pending $x = y$ goal, bring hypotheses with lasso.

- New goal: $y = y + 1$. We need to find contradiction in hypotheses.
- One hypothesis is $parity(y + 1) = parity(y)$, which is false.
- Use dc with $parity(y) = 0$. This causes two instantiations that make proving inconsistency easier.
- P0 works for both branches.

# Proving remaining POs (in `ascending`)

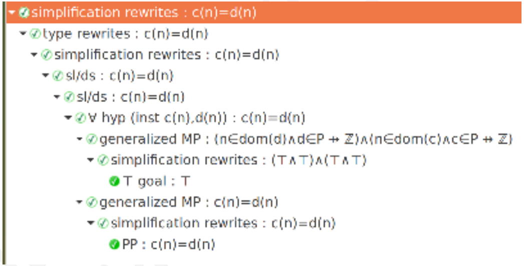
## GRD of $q(n) = p(n)$

- Do lasso.
- Instantiate theorem

$$\forall x, y \cdot y \in \mathbb{N} \land x \in y..y+1 \quad \Rightarrow$$
$$(parity(x) = parity(y) \Leftrightarrow x = y)$$

  with $c(n)$, $d(n)$.

  (Bring it from hypotheses if not among selected hypotheses).
- Note: instantiate the right variable with the right value!
- Invoke P0 for the branches remaining to be proven.



# Proving POs (in `ascending`)

GRD of $\forall m \cdot m \in f^{\sim}[\{n\}] \Rightarrow p(n) \neq p(m)$

Idea: we have to prove that if $p(m) \neq p(f(m))$, then $c(n) \neq c(f(m))$. We have a theorem that says $parity(x) = parity(y) \Leftrightarrow x = y$ when $x \in y..y+1$. So we need $c(n) \in c(f(n))..c(f(n))+1$ to apply it. We add it as a **theorem**, which is immediately proven.

1. Add a new THM: $\forall n \cdot n \in P \setminus \{r\} \Rightarrow c(n) \in c(f(n))..c(f(n))+1$
2. Click on the PO for the undischarged GRD.
3. Introduce the hypothesis $n = f(m)$ (which comes from $m \in f^{-1}[\{n\}]$) with ah and use ML repeatedly.
4. If some subgoal is not proven, bring all the available hypotheses from the Search window and use ML.

# Discharging POs (in `descending`)

- In my case, GRD for $q(n) \neq q(f(n))$ in **descending_nr** remains to be proven.
- It should imply $d(n) \neq d(f(n))$.
- Similar to the previous case.
- Add a symmetrical theorem
  $\forall n \cdot n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1$
- It is immediately proven and it also automatically discharges the pending GRD proof.

# Discharging POs (in `descending`)

- With Rodin 3.8 it may be the case that the invariant

$$\forall n \cdot n \in P \Rightarrow p(n) = parity(c(n)))$$

  is unproven.
- The goal to be proven is

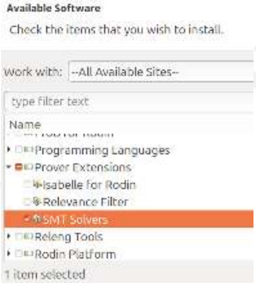
$$1 - p(n) = parity(c(n) + 1)$$

  which is immediate from the definition or *parity* and $c(n)$.
- Just apply lasso, instantiate the definition of *parity* with $c(n)$, and use ML or P0

At this point, all the POs should be discharged.

# Less Manual Work?

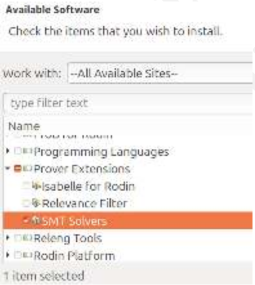
- *Atelier B* provers: developed for (Event-)B, integrated with Rodin.
- Not the most powerful provers.
- Additional provers: Install Software → Work with "– All Available Sites –" → Prover Extensions → SMT Solvers.




- Can often discharge proofs without / with less manual intervention.

# Less Manual Work?

- *Atelier B* provers: developed for (Event-)B, integrated with Rodin.
- Not the most powerful provers.
- Additional provers: Install Software → Work with "– All Available Sites –" → Prover Extensions → SMT Solvers.




- Can often discharge proofs without / with less manual intervention.

- Why not using them before?
  - SMT solvers external → stability not guaranteed?
  - If using SMT Solvers, examples requiring interaction likely too complex for a first contact.
- Install and try the SMT solvers in the examples in this section of the course using less additional theorems / invariants.
- Plugin feeds SMT solvers with "Selected Hypothesis". Possible heuristics:
  - Bring hypotheses to the "Selected" set with lasso, or
  - Select all hypotheses in "Search" tab, add them to "Selected".