# Event B: Modeling and Reasoning with Data Structures[1]

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[1] With material from J. R. Abrial book *Modeling in Event-B: system and software engineering*.

# Strategy

- Data structures involving pointers / references described with relations, functions.
- Specific axioms of these specific data structures give *properties* of the functions that model the data structures.
- These properties are necessary for theorem provers to discharge proofs on data structures.
- Specific forms of these axioms (capturing induction on the data structures) are well-suited to be used in automated proofs.

- We will focus on formalizing:
  - Infinite lists.
  - Finite lists.
  - Infinite trees.
  - Finite trees.

# Infinite lists

- Set $V$ of list nodes.
- Initial node $f$.
- Bijective *next* function.

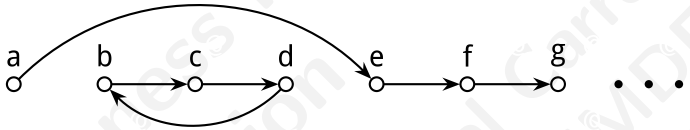
  There is a *next* and a *previous* (with the exception of $f$)

$$\text{axm\_1}: \quad f \in V$$

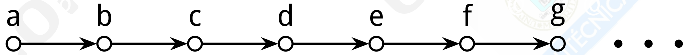
$$\text{axm\_2}: \quad n \in V \rightarrowtail\!\!\!\!\twoheadrightarrow V \setminus \{f\}$$

# Characterizing (and avoiding) cycles

Cycles:




No cycles:



# Characterizing (and avoiding) cycles

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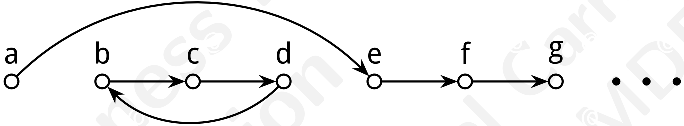
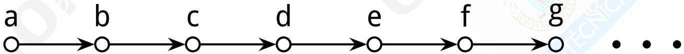



$S = \{b, c, d\}$ $\quad n[S] = \{b, c, d\}$ $\quad S \subseteq n[S]$

No cycles:



# Characterizing (and avoiding) cycles

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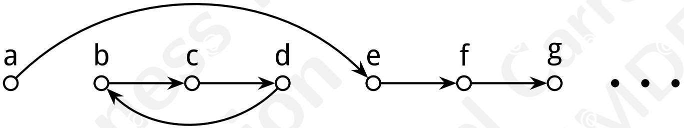


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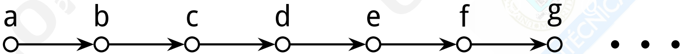



$S = \{b, c, d\}$ $\quad$ $n[S] = \{c, d, e\}$ $\quad$ $S \not\subseteq n[S]$

(for *almost* any $S \subseteq V$)

# Avoiding cycles

- If a list has a cycle, then there is a $S \subseteq V$ s.t. $S \subseteq n[S]$.
- On the other hand, it is always the case that $\varnothing \subseteq n[\varnothing]$.
- So we insist that this is the only case:

$$\text{axm\_3} : \forall S \cdot S \subseteq V \land S \subseteq n[S] \Rightarrow S = \varnothing$$

- It can be used to prove properties in infinite lists.
- We will derive from it an axiom scheme of induction.

## From absence of cycles to induction

- Abscense of cycles: $\forall S \cdot S \subseteq V \wedge S \subseteq n[S] \Rightarrow S = \varnothing$
- $S$ can be written as $S = V \setminus T$, for some $T$
  (For example, $T = V \setminus S$ would work)
- Then:

$$\forall S \cdot S = V \setminus T \wedge \boxed{S \subseteq V} \wedge S \subseteq n[S] \Rightarrow S = \varnothing$$

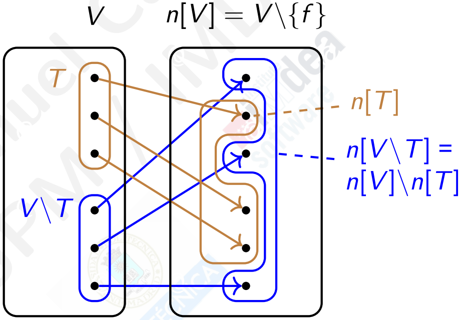
Redundant

- Removing redundant subformula: $\forall S \cdot S = V \setminus T \wedge S \subseteq n[S] \Rightarrow S = \varnothing$
- Let us focus on $S = \varnothing$

# From absence of cycles to induction

Let us simplify $\forall S \cdot S = V \setminus T \wedge S \subseteq n[S] \Rightarrow S = \varnothing$

- If $S = V \setminus T$, then
  $S = \varnothing \ \equiv \ V \setminus T = \varnothing \ \equiv \ V \subseteq T$
- The non-cycle condition then becomes
  $\forall S \cdot S = V \setminus T \wedge S \subseteq n[S] \Rightarrow V \subseteq T$




- Let us focus on $n[S]$
- Since $S = V \setminus T$, $n[S] = n[V \setminus T]$
- Since $n$ is bijective, $n[V \setminus T]$ and $n[T]$ don't intersect (see figure on the right)
- Therefore, $n[V \setminus T] = n[V] \setminus n[T]$

## From absence of cycles to induction

- Since $S = V \setminus T$ and $n[V \setminus T] = n[V] \setminus n[T]$, $S \subseteq n[S]$ becomes $V \setminus T \subseteq n[V] \setminus n[T]$
- Let us simplify that condition
- By definition: $f \in V$ and $f \notin n[V \setminus T]$ ($f$ is not in the range of $n$)
- Since $V \setminus T \subseteq n[V \setminus T]$, $f \notin V \setminus T$
  (because $f \notin n[V \setminus T]$ and $V \setminus T$ is a subset of $n[V \setminus T]$)
- Therefore $f$ must be *subtracted* from $V$ by $T$, and then $f \in T$
- Also by definition, $n[V] = V \setminus \{f\}$.
- So we can rewrite $V \setminus T \subseteq n[V] \setminus n[T]$ as $V \setminus T \subseteq (V \setminus \{f\}) \setminus n[T]$

## From absence of cycles to induction

- Let us simplify $\underbrace{\overbrace{V}^{a} \setminus \overbrace{T}^{b}}_{e} \subseteq \underbrace{\overbrace{(V \setminus \{f\})}^{c} \setminus \overbrace{n[T]}^{d}}_{f}$.
- We know that $f \in V$ and $f \in T$.
- $f$ is not in set ($f$), and then it should not be in ($e$); it is removed by ($b$).
- Then we have to worry about how much is removed by ($b$) and ($d$).
- If ($d$) removes "too much", then ($e$) will be larger.
- I.e., if ($d$) contains an element that is not in ($b$), then ($e$) will contain an element that is not in ($f$).
- Therefore, ($d$) cannot contain elements that are not in ($b$)
  (or, any element in ($d$) must also be in ($b$))
- So the formula simplifies to $n[T] \subseteq T$.

## From absence of cycles to induction

Putting all together, the non-cycle condition becomes

$$\forall S \cdot S = V \setminus T \land f \in T \land n[T] \subseteq T \Rightarrow V \subseteq T$$

If we expand $n[T] \subseteq T$:

$$\text{thm\_2} : \forall T \cdot f \in T \land (\forall x \cdot x \in T \Rightarrow n(x) \in T) \Rightarrow V \subseteq T$$

- $T$ the set of elements with some property $P$: $T = \{x | P(x)\}$
- So the meaning of thm_2 is:
  - If the initial node $f$ has property $P$ ($f \in T$), and
  - For every element with property $P$ ($x \in T$), the next one has this property ($n(x) \in T$), then
  - All elements have property $P$ ($V \subseteq T$).

# Using thm_2 to prove list properties

- We want to prove $P(x)$ for all $x \in V$.
- Elements for which $P$ holds: $T = \{x | x \in V \wedge P(X)\}$.
- We want to prove that $T = V$.
- Since by construction $T \subseteq V$, it is enough to prove $V \subseteq T$.
- We do that by instantiating $T$ in thm_2.

$$
\begin{array}{r}
f \in \{x | x \in V \wedge P(x)\} \qquad\qquad \wedge \\
\forall x \cdot x \in \{x | x \in V \wedge P(x)\} \Rightarrow n(x) \in \{x | x \in V \wedge P(x)\} \quad \Rightarrow \\
V \subseteq \{x | x \in V \wedge P(x)\} \qquad\qquad
\end{array}
$$

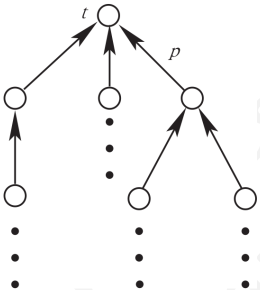
- $f \in \{x | x \in V \wedge P(x)\} \equiv P(f)$.
- Second part equivalent to $\forall x \cdot x \in V \wedge P(x) \Rightarrow P(n(x))$.
- The RHS is equivalent to $\forall x \cdot x \in V \Rightarrow P(x)$.
- Instantiating thm_2 gives a scheme to prove by induction in infinite lists.

# Finite lists

- Basically as infinite lists, but including a last ($l$) element and a different axiom 2:

$$
\begin{aligned}
\text{axm\_4}: &\quad l \in V \\
\text{axm\_5}: &\quad \text{finite}(V) \\
\text{axm\_2}': &\quad n \in V \setminus \{l\} \rightarrowtail\!\!\!\!\twoheadrightarrow V \setminus \{f\} \\
\text{induction}: &\quad \forall T \cdot T \subseteq V \land f \in T \land (\forall x \cdot x \in V \setminus \{l\} \land x \in T \Rightarrow n(x) \in T) \Rightarrow V \subseteq T
\end{aligned}
$$

# Infinite trees




- $t$ is the root.
- $p$ links node with parent (surjection).
- No cycles.

$$\begin{aligned}
\text{axm_1}: &\quad t \in V \\
\text{axm_2}: &\quad p \in V \setminus \{t\} \twoheadrightarrow V \\
\text{axm_3}: &\quad \forall S \cdot S \subseteq p^{-1}[S] \Rightarrow S = \varnothing
\end{aligned}$$

Induction rule:

$$\forall T \cdot t \in T \land p^{-1}[T] \subseteq T \Rightarrow V \subseteq T$$

Instantiation to prove properties:

$$\begin{aligned}
\forall T \cdot \quad & T \subseteq V \land t \in T \land \\
& (\forall x \cdot x \in V \setminus \{t\} \land p(x) \in T \Rightarrow x \in T) \\
& \Rightarrow V \subseteq T
\end{aligned}$$

Note: placement of $p$ in implication is *opposite* w.r.t. $f$ for lists – "direction" of arrows reversed!

# Finite trees

- $t$ is the root.
- $p$ relates every node with its parent.
- $L$ is the set of tree leaves.
- There should not be cycles.

$$
\begin{array}{ll}
\text{axm\_1}: & t \in V \\
\text{axm\_2}: & L \subseteq V \\
\text{axm\_3}: & p \in V \setminus \{t\} \twoheadrightarrow V \setminus L \\
\text{axm\_4}: & \forall S \cdot S \subseteq p^{-1}[S] \Rightarrow S = \varnothing
\end{array}
$$

The induction scheme is as in infinite trees.