Synchronizing Processes on a Tree Network[1]

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[1] Example and several slides from J. R. Abrial book *Modeling in Event-B: system and software engineering*.

Purpose of this lecture

- We will formalize the solution to a problem in distributed computing.
  - Studied in: *W.H.J. Feijen and A.J.M. van Gasteren. On a Method of Multi-programming. Springer Verlag, 1999.*
- Using and updating functions.
- Formalize and prove properties on an interesting structure: a tree.
- Proofs more complex than those seen so far.

As usual:

- Define the informal requirements
- Define the refinement strategy
- Construct the various more and more concrete models

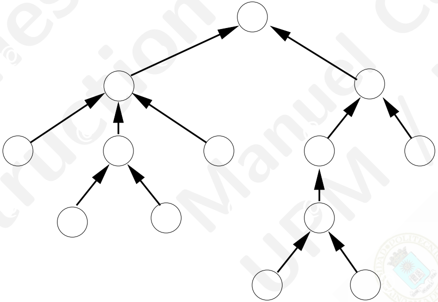
Comparison with previous examples

- Not a transformational system.
  - Not supposed to finish.
  - No final result.
- Not reactive.
  - No *external* world that reacts to system changes.
- Distributed.
  - Different *nodes* act autonomously.
  - With limited information access.
  - However, communication assumed to be reliable.

- Internal concurrency.
  - Every node has concurrent processes.
- Small model: just three events in the last refinement.
- However, proofs and reasoning are comparatively complex.

Requirements

ENV 1	We have a fixed set of processes forming a tree



- Note: they do not need to form a tree from the beginning.
- A set of communicating processes can coordinate to form a tree.

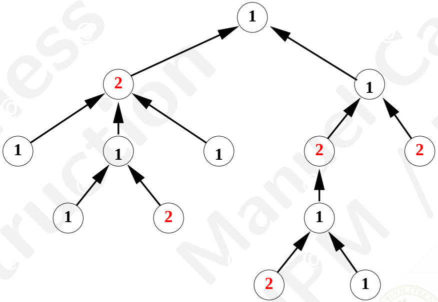
Requirements (Cont.)

- All processes are supposed to execute forever the same code.
- But processes must remain (somewhat) synchronized.
- For this, each process has (initially) one counter.

ENV 2	Each process has a counter, which is a natural number

- A process counter represents its "phase"
(related to the work for which they have to synchronize).
- Difference between any two counters $\leq$ one.
  - Each process is thus at most one phase ahead of the others

Requirements (Cont.)



FUN 3	The difference between any two counters is at most one

Requirements (Cont.)

- Reading the counters

FUN 4	Each process can read the counters of its neighbors only

(*Neighbors* to be understood as *connected by a link*)

- Modifying the counters

FUN 5	The counter of a process can be modified by this process only

Refinement strategy

- Construct abstract initial model dealing with FUN 3 and FUN 5
- Improve design to (partially) take care of FUN 4
- Improve design to better take care of FUN 4
- (Simplify final design to obtain efficient implementation).

FUN 3	The difference between any two counters is at most one

FUN 4	Processes read counters of immediate neighbors only

FUN 5	A process can modify only its counter(s)

Steps

1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

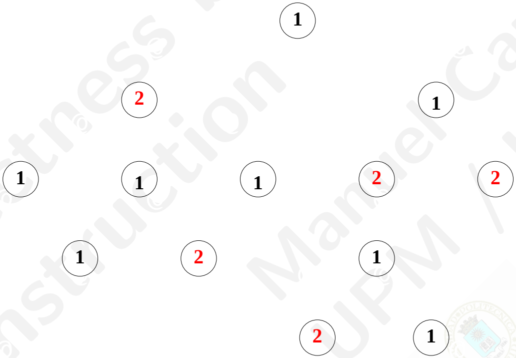
Initial model: the state

- Simplify situation: forget about tree
- We just define the counters and express the main property: FUN 3

FUN 3	The difference between any two counters is at most one

- The initial model is always far more abstract than the final system
- Other requirements are probably not fulfilled

Abstract situation



FUN 3	The difference between any two counters is at most 1

Suggest constants, axioms, variables, invariants for an initial model!

Initial model: the state

carrier set: P

axm0_1: $\text{finite}(P)$

variable: c

inv0_1: $c \in P \rightarrow \mathbb{N}$

inv0_2: $\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{pmatrix}$

- ✓ *Create project* `synch_tree`
- ✓ *Create context* `c0` *with set, axiom*
- ✓ *Create machine* `m0` *with variable, invariants.*

Is that right?

- inv0_2 may be surprising:

$$\mathcal{I}_0 : \forall x, y \cdot x \in P \land y \in P \Rightarrow c(x) \leq c(y) + 1$$

- Is it the same as $\mathcal{I}_1 : \forall i, j \cdot |c(i) - c(j)| \leq 1$?

Proof by double implication.

Let us choose two arbitrary nodes with counters a and b.

- If the invariant holds, then $a \leq b + 1$ and $b \leq a + 1$. From there, $a - b \leq 1$ and $b - a \leq 1$, therefore $|a - b| \leq 1$, and $\mathcal{I}_0 \Rightarrow \mathcal{I}_1$.
- If $|a - b| \leq 1$, then both $a - b \leq 1$ and $b - a \leq 1$. Then, inv0_2 is implied by the intended invariant, and $\mathcal{I}_1 \Rightarrow \mathcal{I}_0$.

Initial model: events

init
$c := P \times \{0\}$

ascending
any n **where**
$n \in P$
$\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
then
$c(n) := c(n) + 1$
end

- Note: any n is logically $\forall n \cdot n \in P \wedge \cdots$
  - $\forall$ can appear in guards.
  - any introduces $\forall$ whose scope is the whole event.
- Intuition: *Any increment that respects difference among nodes can be done*.
- Does not mean all increments are executed: non-determinism!
- Not final state (there is none): action that (hopefully) respects invariant.

✓ *Add initialization, event*

Note: $\times$ is entered with **, any with pull-down menu, "Add event parameter".

Proof of invariant preservation

$c \in P \to \mathbb{N}$ **inv0_1**

$\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{pmatrix}$ **inv0_2**

$n \in P$

$\forall m \cdot (m \in P \Rightarrow c(n) \leq c(m))$ Guards of event ascending

$\vdash$

$$\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ (c \mathbin{\lhd\!\!\!-} \{n \mapsto c(n)+1\})(x) \leq (c \mathbin{\lhd\!\!\!-} \{n \mapsto c(n)+1\})(y) + 1 \end{pmatrix}$$

$\Uparrow$

Modified invariant **inv0_2**

In Rodin: automatic; if not, repeatedly apply lassoing, p0 or m0.

Model so far

CONTEXT c0
SETS
  P
AXIOMS
  axm1: $finite(P)$
END

MACHINE m0
SEES c0
VARIABLES
  c
INVARIANTS
  inv1: $c \in P \rightarrow \mathbb{N}$
  inv2: $\forall x, y \cdot x \in P \land y \in P \Rightarrow c(x) \leq 1 + c(y)$
EVENTS
Initialisation
  begin
    act1: $c := P \times \{0\}$
  end
Event ascending ⟨ordinary⟩ $\widehat{=}$
  any
    n
  where
    grd12: $n \in P$
    grd11: $\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
  then
    act11: $c(n) := c(n) + 1$
  end

Problem with the current event

ascending
  any n **where**
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
  then
    $c(n) := c(n) + 1$
  end

What requirement is this event breaking?

FUN 2	Each node can read the counters of its immediate neighbors only

Steps

1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

First refinement: (partially) solving the problem

- Introduce a designated process r.
- Assume that counter of r always minimal

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

- Rationale:
  - We only synchronize with r — not compliant, but communication restricted.
  - Helps ensure that difference between any two nodes $\leq$ one.
    - If inv0_1: $\forall x, y \cdot x \in P \land y \in P \Rightarrow c(x) \leq c(y) + 1$, then $|c(r) - c(m)| \leq 1$ for any m.
    - If $c(r) \leq c(m)$, then $c(m) = c(r)$ or $c(m) = c(r) + 1$ for any m.
    - Then $|c(m) - c(n) \leq 1|$, for any m, n (will be proved).
- Treat this property as a new (temporary) invariant.

✓ *Extend `c0` into `c1` (left pane, right click, "Extend"), add constant r, axiom $r \in P$*
✓ *Refine `m0` into `m1` (left pane, right click, "Refine"), add new invariant*
✓ *`m0` should "see" `c1`*

First refinement: proposal for the event refinement

We simplify the guard

(abstract-)ascending	(concrete-)ascending
any n **where** $n \in P$ $\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$ **then** $c(n) := c(n) + 1$ **end**	**any** n **where** $n \in P$ $c(n) = c(r)$ **then** $c(n) := c(n) + 1$ **end**

- Note: if $c(r)$ minimal, $c(n) < c(r)$ impossible; therefore $c(n) = c(r)$

  ✓ *Change "extended" to "not extended", change guard*

- We have then to prove guard strengthening.

Guard strengthening

$c \in P \rightarrow \mathbb{N}$ **inv0_1**

$\forall x, y \cdot \left(\begin{array}{l} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{array} \right)$ **inv0_2**

$\forall m \cdot (\, m \in P \;\Rightarrow\; \boxed{c(r)} \leq c(m)\,)$ **new invariant**

$n \in P$

$\boxed{c(n) = c(r)}$

Guards of concrete event ascending

$\vdash$

$n \in P$

$\forall m \cdot (\, m \in P \Rightarrow \boxed{c(n)} \leq c(m)\,)$

Guards of abstract event ascending

In Rodin: lasso + p0

✓ *Go to the proving perspective, discharge proof*

Model so far

`inv1` not discharged.

CONTEXT c1
EXTENDS c0
CONSTANTS
    r
AXIOMS
    axm1: $r \in P$
END

MACHINE m1
REFINES m0
SEES c1
VARIABLES
    c
INVARIANTS
    inv1: $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$
EVENTS
Initialisation ⟨extended⟩
    begin
        act1: $c := P \times \{0\}$
    end
Event ascending ⟨ordinary⟩ $\widehat{=}$
refines ascending
    any
        n
    where
        grd1: $n \in P$
        grd2: $c(r) = c(n)$
    then
        act1: $c(n) := c(n) + 1$
    end
END

Pending problems

```
ascending
  any n where
    n ∈ P
    c(n) = c(r)
  then
    c(n) := c(n) + 1
  end
```

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

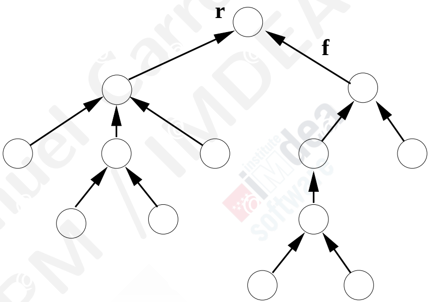
1. Prove that new "invariant" is preserved by the event.
2. The guard of the event still does not fulfill requirement FUN 4.

FUN 4	Each node can read the counters of its immediate neighbors only

- Problem 1 solved in this refinement
- Problem 2 solved later

First refinement: defining the tree

- Tree: root r and "pointer" f from each node in $P \setminus \{r\}$ to every node's parent.
- Plus some additional properties and inference rules.
- Reminder: sets, relations, functions, specific data structures and their inference rules.
- Note: constructing a tree ($\equiv$ root / leader + links) is a classical problem in distributed systems.
- Can also be tackled using Event B.



Invariant: we have a condition involving nodes in pairs and we need a condition that relates a single node r with all the others.

Update model

✓ *Add to `c1` (note f is $\twoheadrightarrow$, written `-->>`)*

- Constant f.
- Axioms:

$$L \subseteq P$$
$$f \in P \setminus \{r\} \twoheadrightarrow P \setminus L$$
$$\forall S \cdot S \subseteq f^{-1}[S] \Rightarrow S = \varnothing$$

- f^{-1} is written `f~`.
- $\twoheadrightarrow$: f defined for all $P \setminus \{r\}$ and *arrives* to every element in $P \setminus L$.

Minimal counter at the root

- Minimality of counter at the root

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

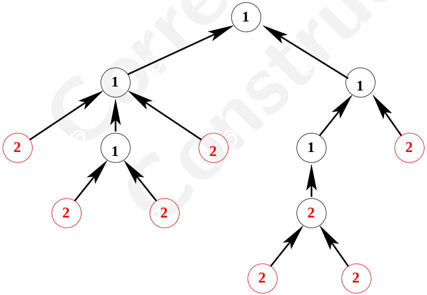
  relates $c(r)$ with $c(m)$ for every m.
- Events change local values and consult neighbouring values.
- We can (easily) prove properties relating neighbouring nodes.
- How can we relate local properties with global properties?

Minimal counter at the root

- We define a weaker, local invariant first.
- The counter at every node is not greater than its descendants:

$$\text{inv1\_1} : \forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$$

✓ *Add it to* `m1`



Rationale (advancing the algorithm)

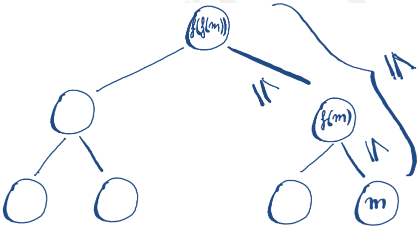
- Assume we can update the tree keeping a maximum difference between neighbors.
- Will be helpful to prove $c(r) \leq c(m)$.

Minimal counter at the root: induction

We need to *extend* the local property

$$\forall m \cdot m \in P \backslash \{r\} \Rightarrow c(f(m)) \leq c(m)$$

to the whole tree.



- Start with leaves $l \in L$.
- Prove that for any l, $c(f(l)) \leq c(l)$, then $c(f(f(l))) \leq c(f(l)) \leq c(l), \ldots$
- Work upwards towards root r.

OR

- Start with r.
- Prove that for all m s.t. $r = f(m)$, $c(r) \leq c(m)$.
  m is a child of r
- Then, for all m' s.t. $m = f(m')$, $c(m) \leq c(m')\ldots$
- And so on towards the leaves.

Minimal counter at the root: induction

- Induction: difficult for theorem provers to do on their own.
  - Needs to identify base case, property to use for induction.
- Then, proving property given base case & inductive step within theorem provers' capabilities.
- In Rodin: needs adding induction scheme:

✓ *Add to* `c1`*:*

$\forall S \cdot S \subseteq P \land r \in S \land (\forall n \cdot n \in P \setminus \{r\} \land f(n) \in S \Rightarrow n \in S) \Rightarrow P \subseteq S$

✓ *Tip:* `Ctrl-Enter` *breaks text in input box in separate lines.*

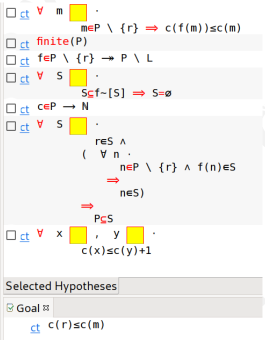
- Instantiating it with the property to prove, expressed as a set: `{x | x ∈ P ∧ c(r) ≤ c(x)}` (next slide)

✓ *In* `m1`*: ensure you have* inv1_1: $\forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$

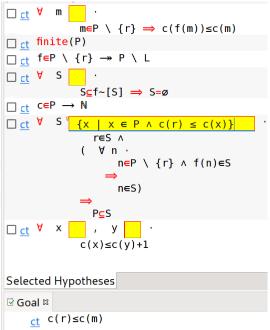
✓ *Ensure* thm1_1: $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$ *below invariant, marked as theorem*

Induction in Rodin: instantiation

- Double click in the unproved theorem (left pane).
- Switch to prover view, lasso.
- Locate induction axiom.
- Enter
  {x | x ∈ P ∧ c(r) ≤ c(x)}.
- Return and p0.
- The theorem should be proved now.



⇒



Invariant inv1_1 not yet proved. Requires order between parent and children $c(f(m)) \leq c(m)$ that ascending cannot guarantee: guard $c(r) = c(n)$ allows updates in arbitrary order. Will enforce through more local comparison.

More local comparison

- Nodes with difference $\leq$ one from r.
- When can we update looking locally?

ascending

any n where

$n \in P$

$c(r) = c(n)$

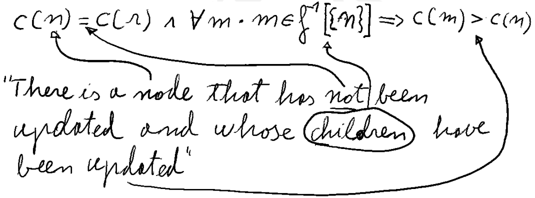
$\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$

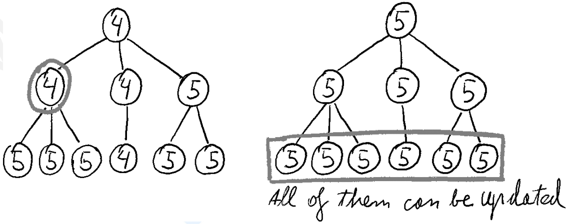
then

c(n) := c(n) + 1

end

Ensure inv1_1 is preserved: double click, prover view, lasso, p0 should do it.

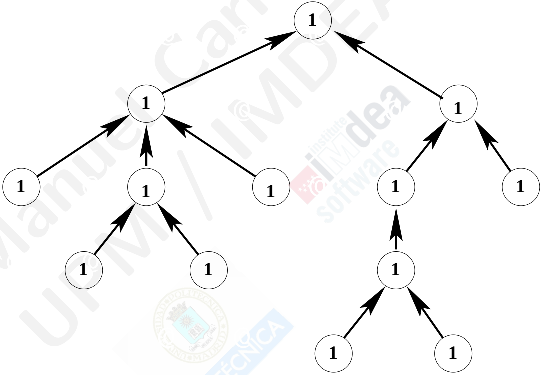




How it is expected to work

Update order restricted:

- **Before:** any node whose counter is equal to the root (the one with the minimum).
- **Now:** only those nodes whose counters are, in addition, smaller than all its descendants.
- Updates will go in waves towards the root.



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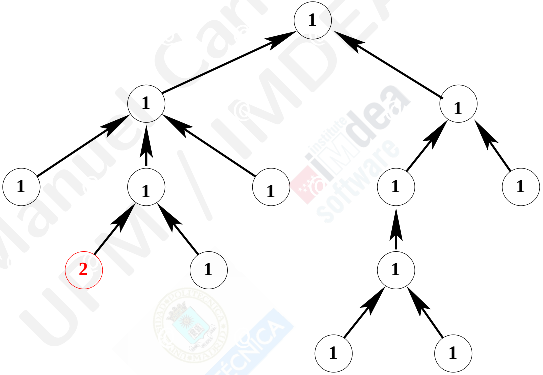

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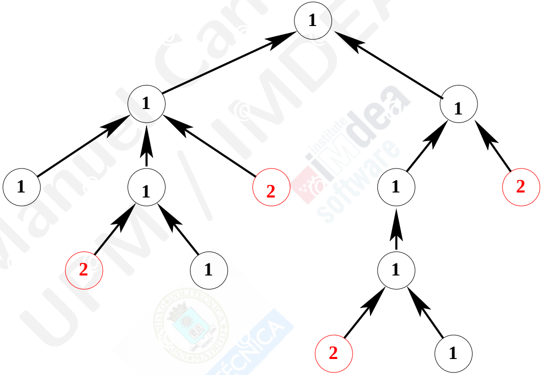

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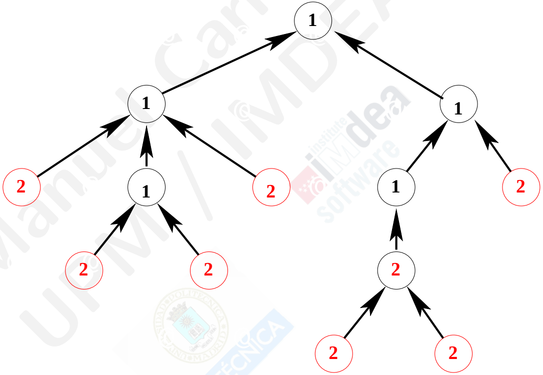

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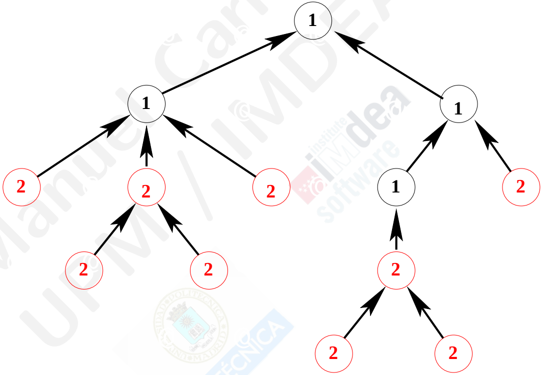

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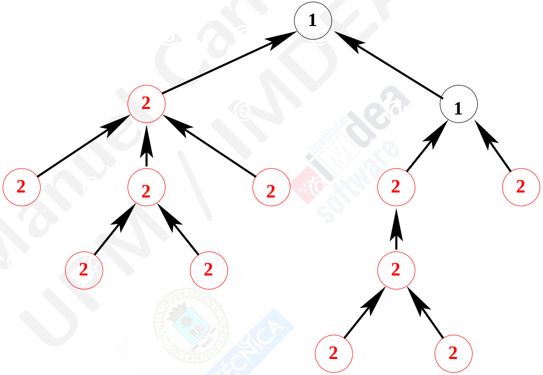

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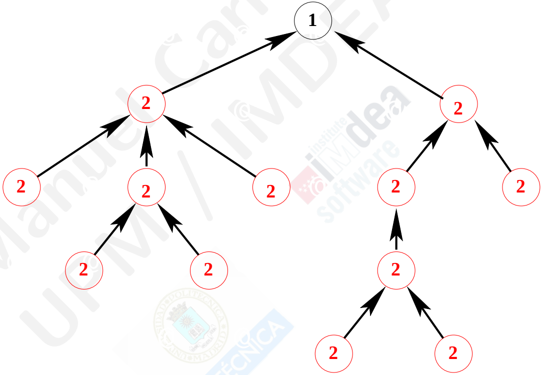

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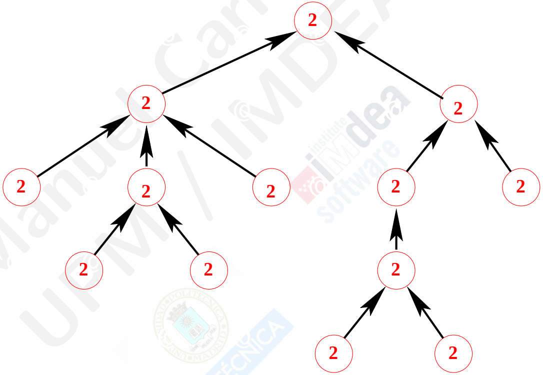

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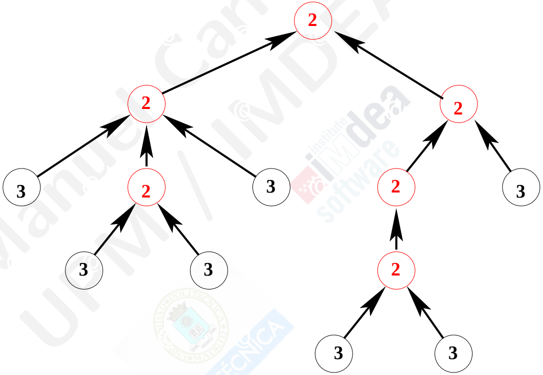

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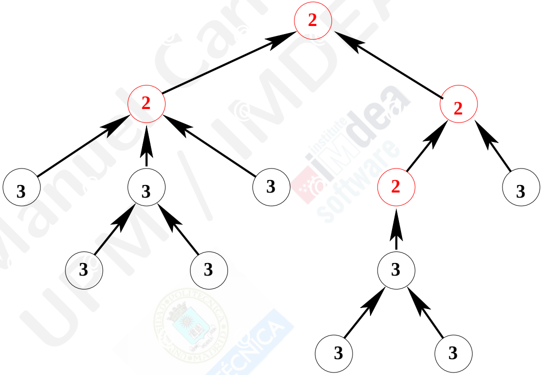

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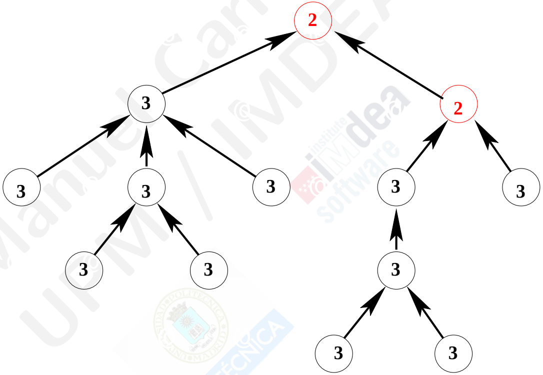

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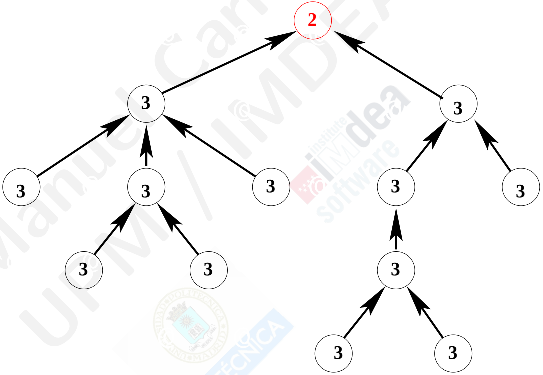


Neighborhood checking

FUN 4	Each process can read the counters of its immediate neighbors only

- $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$ uses only local comparisons.
- $c(r) = c(n)$ uses non-local comparisons.
- We will tackle that in the next refinement.

Model so far

CONTEXT c1
EXTENDS c0
CONSTANTS
r
f
L
AXIOMS
axm1: $r \in P$
axm3: $L \subseteq P$
Leaves
axm2: $f \in P \setminus \{r\} \twoheadrightarrow P \setminus L$
axm4: $\forall S \cdot S \subseteq f^{-1}[S] \Rightarrow S = \varnothing$
axm5:
$\forall S \cdot S \subseteq P \wedge$
$r \in S \wedge$
$(\forall n \cdot n \in P \setminus \{r\} \wedge f(n) \in S \Rightarrow n \in S)$
$\Rightarrow$
$P \subseteq S$
END

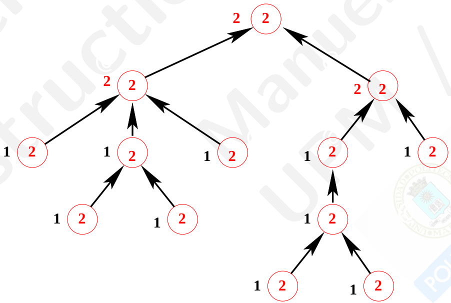
MACHINE m1
REFINES m0
SEES c1
VARIABLES
c
INVARIANTS
inv1: $\forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$
inv2: ⟨theorem⟩ $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$
EVENTS
Initialisation ⟨extended⟩
begin
act1: $c := P \times \{0\}$
end
Event ascending ⟨ordinary⟩ $\widehat{=}$
refines ascending
any
n
where
grd1: $n \in P$
grd2: $c(r) = c(n)$
grd3: $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
then
act1: $c(n) := c(n) + 1$
end
END

Steps

1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

Second refinement

- Replace the guard $c(r) = c(n)$.
- Processes must be aware when this situation does occur.
- Add second counter $d(\cdot)$ to each node to capture value of $c(r)$.



Second refinement: the state

carrier set: P

constants: r, f

variables: c, d

Invariant **inv2_2** is as **inv0_2**

inv2_1: $d \in P \rightarrow \mathbb{N}$

inv2_2: $\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ d(x) \leq d(y) + 1 \end{pmatrix}$

d has an overall property similar to c:

$\forall x, y \cdot x \in P \wedge y \in P \Rightarrow c(x) \leq c(y) + 1$

- d will capture the value of $c(r)$.
- It will be updated in a downward wave.

✓ *Refine* `m1` *into* `m2`

✓ *Add variable* `d` *and invariants*

Updating d

This refinement captures:

- The existence of d.
- How its update can proceed not to break its invariant.

Event descending
  any n where
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow d(n) \leq d(m)$
  then
    $d(n) := d(n) + 1$
  end

✓ *Add event to* m2

✓ *Initialize* d *to 0 (copy the initialization of* c*)*

Steps

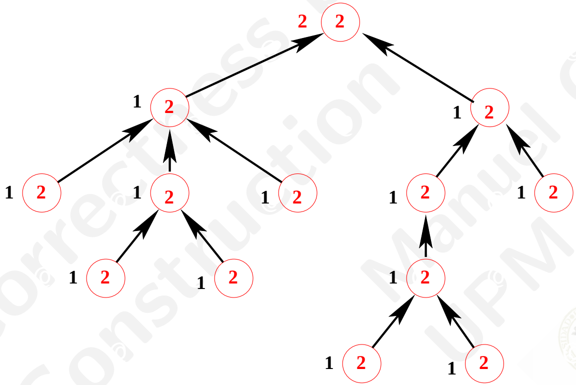
1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

Third refinement

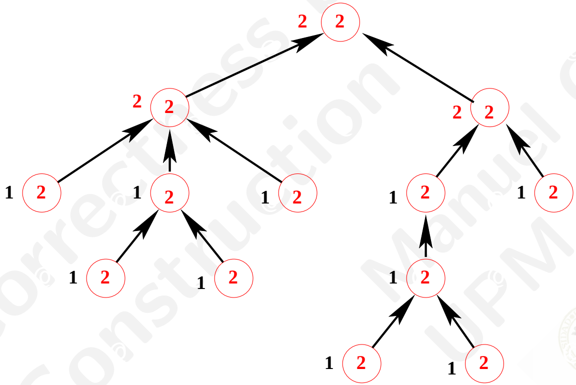
- We extend the invariant of counter d.
- We establish the relationship between both counters c and d.
  - This will allow us to refine event ascending
- We construct the descending wave (by refining event descending).
- Remark: this is the most difficult refinement.

✓ *Refine `m2` into `m3`*

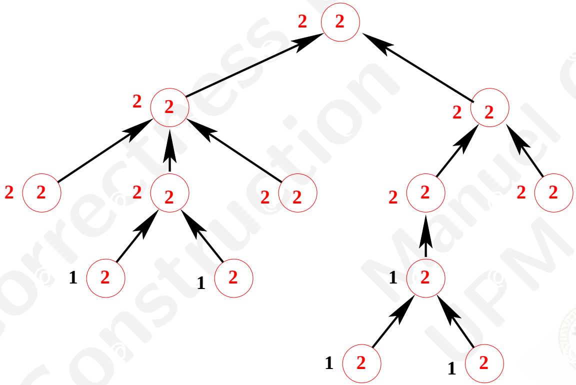
Idea behind third refinement



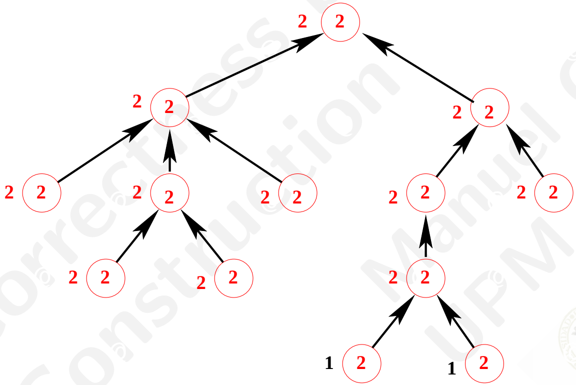
Idea behind third refinement



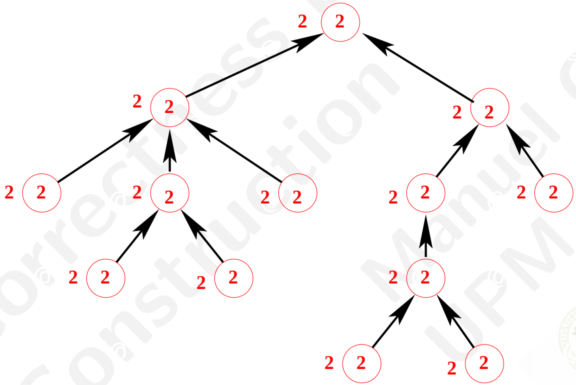
Idea behind third refinement



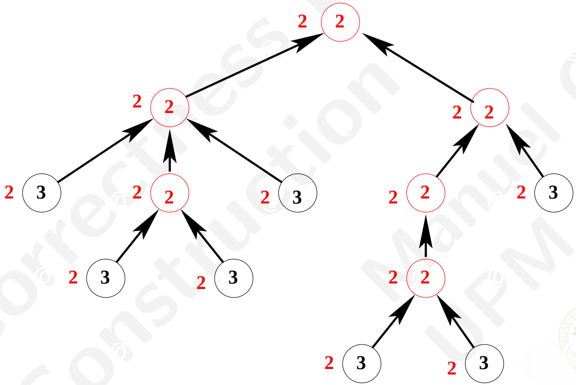
Idea behind third refinement



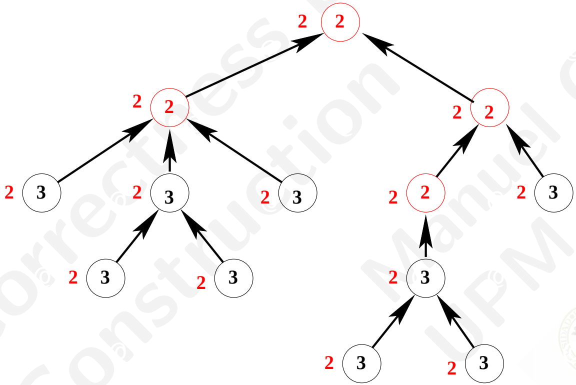
Idea behind third refinement



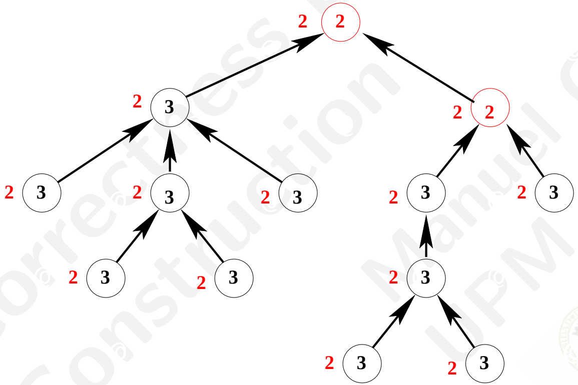
Idea behind third refinement



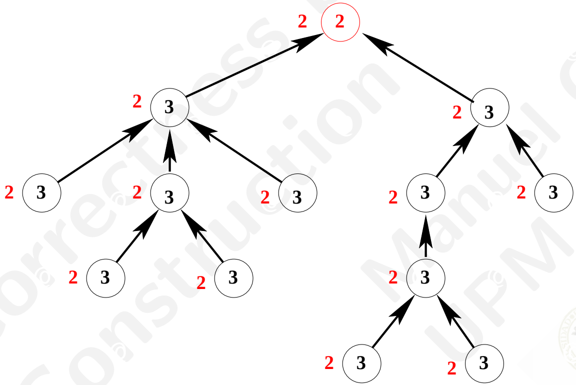
Idea behind third refinement



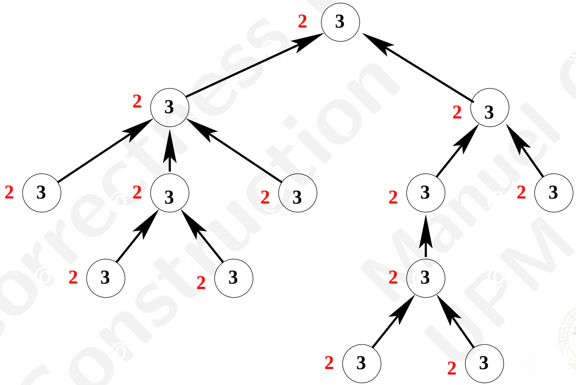
Idea behind third refinement



Idea behind third refinement



Idea behind third refinement



State and invariants

- Recall local condition for c:

$$\mathrm{inv1\_1}: \forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$$

Every node's counter is smaller than or equal to its children's.

- Local condition for d is similar:

$$\mathrm{inv3\_1}: \forall m \cdot m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m))$$

Every node's counter is smaller than or equal to its parent (if it has a parent). This is what makes the wave *descending*.

- inv3_1 and tree induction proves that the root has the highest value of $d(\cdot)$:

$$\mathrm{thm3\_1}: \forall n \cdot n \in P \Rightarrow d(n) \leq d(r)$$

(remember: root had the smallest value of $c(\cdot)$)

Proving theorem and invariant

- ✓ *Add to* `m3`*:*

$$inv3\_1: \quad \forall m \cdot m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m))$$

$$thm3\_1: \quad \forall n \cdot n \in P \Rightarrow d(n) \leq d(r)$$

- ✓ *Mark the latter as theorem*
- ✓ *Double click on the PO for THM*
- ✓ *Go to proving perspective; locate induction axiom*
- ✓ *Instantiate with* $\{x | x \in P \land d(x) \leq d(r)\}$*, invoke p0*
- ✓ *That should prove* $thm3\_1$
- ✓ $inv3\_1$ *cannot be proved yet - reasons similar to* c*.*

We will deal with that later

Refining *ascending*

Event (abstract−)ascending
  any n where
    $n \in P$
    $c(n) = c(r)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  then
    $c(n) := c(n) + 1$
  end

Event (concrete−)ascending
  any n where
    $n \in P$
    $c(n) = d(n)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  then
    $c(n) := c(n) + 1$
  end

- Downward wave d will eventually propagate $d(r)$.
  ✓ *Change event guard in* `m3`
- Need to prove guard strengthening.
- We cannot. c and d unrelated so far!
  ✓ *Relate c and d:* inv3_2 : $d(r) \leq c(r)$
- If needed: proving perspective, lasso + p0 proves strengthening.

ascending: only local comparisons now!

Refining *descending*

- A different case.
- Two situations raise a change of d:
  1. For a non-root node: parent's d change.
  2. For the root node: $c(r)$ changes.
- Different guards.
- We will prepare the events to be edited.

✓ *Change (concrete) descending event to non-extended*
✓ *Left click on circle to left of name to select*
`Ctrl-C` to copy, `Ctrl-V` to paste

✓ *Rename first event as descending_nr.*
✓ *Rename second event as descending_r.*

Refining *descending*: the non-root case

Event (abstract−)descending
any n where
$n \in P$
$\forall m \cdot m \in N \Rightarrow d(n) \leq d(m)$
then
$d(n) := d(n) + 1$
end

Event (concrete−)descending
any n where
$n \in P \setminus \{r\}$
$d(n) \neq d(f(n))$
then
$d(n) := d(n) + 1$
end

✓ *Update guards*

(Note: Rodin $\geq$ 3.6 seems to prove strengthening automatically; previous versions needed additional steps [in next slide])

Proving guard strengthening

Note: the steps below do not seem to be necessary in Rodin 3.6 with the Atelier B provers installed. Strengthening is proven automatically.

$$n \in P \setminus \{r\}, d(n) = d(f(n)), m \in P \vdash d(n) \leq d(m)$$

We need some magic mushrooms to help the provers:

$$\text{thm3\_2}: \quad \forall n \cdot n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1$$

$$\text{thm3\_3}: \quad \forall n \cdot n \in P \Rightarrow d(r) \in d(n)..d(n)+1$$

thm3_2 downward wave, parent is at most one more than children (when it has just been increased)

thm3_3 special case for root (the first one to be increased)

Refining *descending* (Cont. — the root case.)

Event (abstract−)descending
  any n where
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow d(n) \leq d(m)$
  then
    $d(n) := d(n) + 1$
  end

Event (concrete−)descending
  refines
    descending
  when
    $d(r) \neq c(r)$
  with
    n: $n = r$
  then
    $d(r) := d(r) + 1$
  end

✓ *Click on circle left of param. n, delete*

- Parameter n disappeared!
- Substitute *(witness)* for GRD, SIM.
- We are particularizing for r.

- Similar to gluing invariant!
- Note with label: specific Rodin idiom.
- Need to prove
  $d(r) \neq c(r), m \in P \vdash d(r) \leq d(m)$

Finishing proofs

The technique in this slide was necessary for Rodin versions previous to 3.6. For Rodin 3.6 onwards, it seems that it is not necessary. Skip to the next slide.

I needed two more magic pills:

$$\text{inv3\_3}: \quad \forall n \cdot n \in P \Rightarrow c(n) \in d(n)..d(n)+1 \quad \text{To prove GRD}$$

$$\text{thm3\_4}: \quad \forall n \cdot n \in P \Rightarrow c(r) \in d(n)..d(n)+1 \quad \text{To prove inv3\_3}$$

Plus, if not added before:

$$\text{thm3\_2}: \quad \forall n \cdot n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1$$

$$\text{thm3\_3}: \quad \forall n \cdot n \in P \Rightarrow d(r) \in d(n)..d(n)+1$$

After this, the invariant can be proved with a combination of several steps:

- Apply lasso.
- Instantiate $\forall n \cdot c(r) \in d(n)..d(n)+1$ (which relates c and d) with n.
- Remove $\in$ in goal ($c(n) \in d(n)+1..d(n)+1+1$) to create inequalities.
- Do P0 in $c(n) \leq d(n)+1+1$ goal.
- Note that only possibility to prove is $d(n) = c(n)$.
- Do case distinction with $d(n) = c(n)$,
- Apply ML to the subgoals.

Finishing proofs

This strategy is necessary with Rodin 3.6 and 3.7.

An additional invariant is necessary to prove GRD of descending_r:

$$\text{inv3\_3}: \quad \forall n \cdot n \in P \Rightarrow c(n) \in d(n)..d(n)+1$$

After adding it, GRD is immediately proven. However, the invariant remains unproven. It can be proved with the following steps:

- Apply lasso.
- Remove $\in$ in goal $c(n) \in d(n)+1..d(n)+1+1$ to transform it into inequalities that can be proven separately.
- Use ml or p0 for the goal $c(n) \leq d(n)+1+1$.
- For $d(n)+1 \leq c(n)$, do case distinction:
  - Either with $d(n) = c(n)$, or
  - with $d(n)+1 = c(n)$

  and ML to the subgoals.

Third refinement: invariants

inv3_1: $\forall m \cdot (\, m \in P \setminus \{r\} \;\Rightarrow\; d(m) \leq d(f(m))\,)$

inv3_2: $d(r) \leq c(r)$

inv3_3: $\forall n \cdot (\, n \in P \;\Rightarrow\; c(n) \;\in\; d(n) \mathrel{..} d(n)+1\,)$

thm3_1: $\forall m \cdot (\, m \in P \;\Rightarrow\; d(m) \leq d(r)\,)$

thm3_2: $\forall n \cdot (\, n \in P \setminus \{r\} \;\Rightarrow\; d(f(n)) \;\in\; d(n) \mathrel{..} d(n)+1\,)$

thm3_3: $\forall n \cdot (\, n \in P \;\Rightarrow\; d(r) \;\in\; d(n) \mathrel{..} d(n)+1\,)$

thm3_4: $\forall n \cdot (\, n \in P \;\Rightarrow\; c(r) \;\in\; d(n) \mathrel{..} d(n)+1\,)$

Third refinement: events

Event descending_r
  when
    $d(r) \neq c(r)$
  with
    n: $n = r$
  then
    $d(r) := d(r) + 1$
  end

Event descending_nr
  any n where
    $n \in P \setminus \{r\}$
    $d(n) \neq d(f(n))$
  then
    $d(n) := d(n) + 1$
  end

Event ascending
  any n where
    $n \in P$
    $c(n) = d(n)$
    $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$
  then
    $c(n) := c(n) + 1$
  end

Steps

1. Initial model: all nodes access the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

Observation

- The difference among counters is at most one.
  - That has been proven by construction.
- In the guards, we only care whether they are equal or not.
- For this, we only need parity!

$$a, b \in \mathbb{N} \wedge |a - b| \leq 1 \Rightarrow (a = b \Leftrightarrow parity(a) = parity(b))$$

- We will prove that this is a valid refinement.

✓ *Extend context* `c1` *into* `c2`

✓ *Refine* `m3` *into* `m4`

✓ `m4` *should see* `c2`

Formalizing parity

- We replace the counters by their parities
- we add the constant $\color{red}{parity}$

carrier set: P

constants: $r, f, parity$

axm4_1: $parity \in \mathbb{N} \rightarrow \{0, 1\}$

axm4_2: $parity(0) = 0$

axm4_2: $\forall x \,.\, (\, x \in \mathbb{N} \;\Rightarrow\; parity(x + 1) = 1 - parity(x)\,)$

✓ *Add parity and axioms to* c2. *Note: parity is a function!*
✓ *Need some clicking (dom to* $\mathbb{N}$ *+ ML) to prove WD*

The definitions that replace $c(\cdot)$ and $d(\cdot)$

- We replace c and d by p and q

variables: p, q

inv4_1: $p \in P \rightarrow \{0, 1\}$

inv4_2: $q \in P \rightarrow \{0, 1\}$

inv4_3: $\forall n \,.\, (\, n \in P \;\Rightarrow\; p(n) = parity(c(n))\,)$

inv4_4: $\forall n \,.\, (\, n \in P \;\Rightarrow\; q(n) = parity(d(n))\,)$

✓ *Do it in m4. Note the gluing invariants! p and q really syntactic sugar.*
✓ *Remove variables c and d. Not accessed / updated in this refinement!*
✓ *Initialize p and q, remove initializations for c and d.*

New events: counters replaced by parity

ascending
any n **where**
$n \in P$
$p(n) = q(n)$
$\forall m \cdot (\ m \in f^{-1}[\{n\}] \ \Rightarrow \ p(m) \neq p(n)\ )$
then
$p(n) := 1 - p(n)$
end

descending_1
any n **where**
$n \in P \setminus \{r\}$
$q(n) \neq q(f(n))$
then
$q(n) := 1 - q(n)$
end

descending_2
when
$p(r) \neq q(r)$
then
$q(r) := 1 - q(r)$
end

Proving remaining POs (in `ascending`)

GRD of $q(n) = p(n)$

- The essence of the pending GRD proof is $\ldots, q(n) = p(n) \vdash c(n) = d(n)$.
- Depends on proving $parity(a) = parity(b) \Rightarrow a = b$.
- Holds in specific cases (if $|a - b| \leq 1$).
- But theorem provers unable to apply / deduce that property.
- Needs to be stated explicitly:

$$\forall x, y \cdot y \in \mathbb{N} \land x \in y..y+1 \Rightarrow$$
$$(parity(x) = parity(y) \Leftrightarrow x = y)$$

- We could make it axiom, but it can be proven as theorem (better!).

Proving remaining POs (in `ascending`)

Proving new THM in c2

- We need to deal with Well Definedness and the theorem itself.

  WD: Removing dom in goal + P0 takes care of it (if WD is to be discharged).

  THM: Adding hypothesis + case distinction works. See below.

- $\Longleftrightarrow$: split in two implications. One is proven.
- For the other: introduce ah with possible values of x: $x = y \vee x = y + 1$ (because $x \in y..y + 1$ among the hypotheses).
- Prove new hypothesis with ml.
- For the pending $x = y$ goal, bring hypotheses with lasso.

- New goal: $y = y + 1$. We need to find contradiction in hypotheses.
- One hypothesis is $parity(y + 1) = parity(y)$, which is false.
- Use dc with $parity(y) = 0$. This causes two instantiations that make proving inconsistency easier.
- P0 works for both branches.

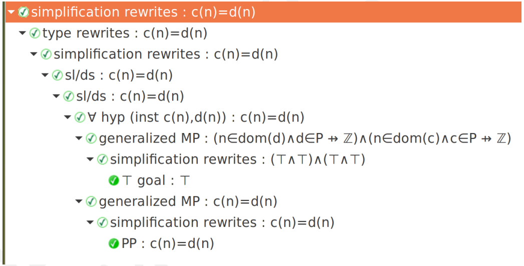
Proving remaining POs (in `ascending`)

GRD of $q(n) = p(n)$

- Do lasso.
- Instantiate theorem

$$\forall x, y \cdot y \in \mathbb{N} \land x \in y..y+1 \quad \Rightarrow$$
$$(parity(x) = parity(y) \Leftrightarrow x = y)$$

  with $c(n)$, $d(n)$.
  (Bring it from hypotheses if not among selected hypotheses).
- Note: instantiate the right variable with the right value!
- Invoke P0 for the branches remaining to be proven.



Proving POs (in ascending)

GRD of $\forall m \cdot m \in f^{\sim}[\{n\}] \Rightarrow p(n) \neq p(m)$

Idea: we have to prove that if $p(m) \neq p(f(m))$, then $c(n) \neq c(f(m))$. We have a theorem that says $parity(x) = parity(y) \Leftrightarrow x = y$ when $x \in y..y+1$. So we need $c(n) \in c(f(n))..c(f(n))+1$ to apply it. We add it as a **theorem**, which is immediately proven.

1. Add a new THM: $\forall n \cdot n \in P \setminus \{r\} \Rightarrow c(n) \in c(f(n))..c(f(n))+1$
2. Click on the PO for the undischarged GRD.
3. Introduce the hypothesis $n = f(m)$ (which comes from $m \in f^{-1}[\{n\}]$) with ah and use ML repeatedly.
4. If some subgoal is not proven, bring all the available hypotheses from the Search window and use ML.

Discharging POs (in `descending`)

- In my case, GRD for $q(n) \neq q(f(n))$ in **descending_nr** remains to be proven.
- It should imply $d(n) \neq d(f(n))$.
- Similar to the previous case.
- Add a symmetrical theorem
  $\forall n \cdot n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1$
- It is immediately proven and it also automatically discharges the pending GRD proof.

Discharging POs (in `descending`)

- With Rodin 3.8 it may be the case that the invariant

$$\forall n \cdot n \in P \Rightarrow p(n) = parity(c(n)))$$

  is unproven.
- The goal to be proven is

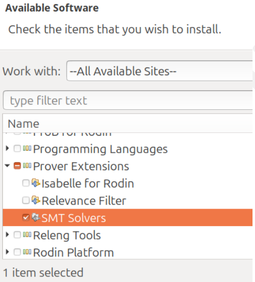
$$1 - p(n) = parity(c(n) + 1)$$

  which is immediate from the definition or *parity* and $c(n)$.
- Just apply lasso, instantiate the definition of *parity* with $c(n)$, and use ML or P0

At this point, all the POs should be discharged.

Less Manual Work?

- *Atelier B* provers: developed for (Event-)B, integrated with Rodin.
- Not the most powerful provers.
- Additional provers: Install Software → Work with "– All Available Sites –" → Prover Extensions → SMT Solvers.



- Can often discharge proofs without / with less manual intervention.

- Why not using them before?
  - SMT solvers external → stability not guaranteed?
  - If using SMT Solvers, examples requiring interaction likely too complex for a first contact.
- Install and try the SMT solvers in the examples in this section of the course using less additional theorems / invariants.
- Plugin feeds SMT solvers with "Selected Hypothesis". Possible heuristics:
  - Bring hypotheses to the "Selected" set with lasso, or
  - Select all hypotheses in "Search" tab, add them to "Selected".