Event B: Modeling and Reasoning with Data Structures[1]

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[1] Theory, text, examples borrowed from J. R. Abrial: see http://wiki.event-b.org/index.php/Event-B_Language

Strategy

- Data structures involving pointers formalized with relations, functions.
- Specific axioms of these specific data structures give *properties* of the functions that model the data structures.
- These properties are necessary for theorem provers to discharge proofs on data structures.
- Specific forms of these axioms (capturing induction on the data structures) are well-suited to be used in automated proofs.

- We will focus on formalizing:
  - Infinite lists.
  - Finite lists.
  - Infinite trees.
  - Finite trees.

Infinite lists

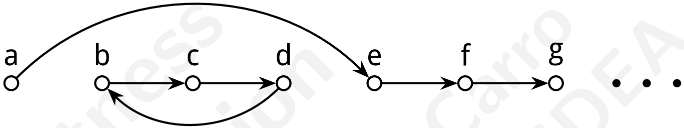
- Set V of list nodes.
- Initial node f.
- Bijective *next* function

axm_1 : $f \in V$

axm_2 : $n \in V \rightarrowtail\!\!\!\!\twoheadrightarrow V \setminus \{f\}$

Characterizing (and avoiding) cycles

Cycles:



$S = \{b, c, d\}$ $n[S] = \{b, c, d\}$ $S \subseteq n[S]$

No cycles:

$S = \{b, c, d\}$ $n[S] = \{c, d, e\}$ $S \not\subseteq n[S]$

(for *almost* any $S \subseteq V$)

Avoiding cycles

- If a list has a cycle, then there is a $S \subseteq V$ s.t. $S \subseteq n[S]$.
- On the other hand, it is always the case that $\varnothing \subseteq n[\varnothing]$.
- So we insist that this is the only case:

$$\text{axm\_3} : \forall S \cdot S \subseteq V \land S \subseteq n[S] \Rightarrow S = \varnothing$$

- It can be used to prove properties in infinite lists.
- We will derive from it an axiom scheme of induction.

From absence of cycles to induction

- Abscense of cycles: $\forall S \cdot S \subseteq V \wedge S \subseteq n[S] \Rightarrow S = \varnothing$
- S can be written as $S = V \setminus T$, for some T
- Then:

$$\forall S \cdot S = V \setminus T \wedge \boxed{S \subseteq V} \wedge S \subseteq n[S] \Rightarrow S = \varnothing$$

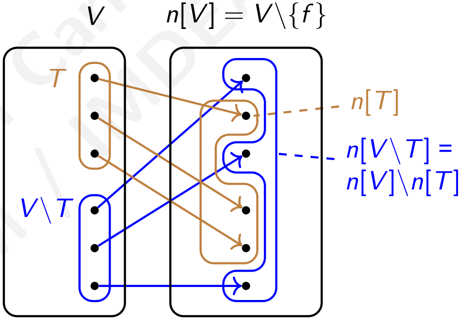
Redundant

- Removing redundant subformula: $\forall S \cdot S = V \setminus T \wedge S \subseteq n[S] \Rightarrow S = \varnothing$
- Let us focus on $S = \varnothing$

From absence of cycles to induction

Let us simplify $\forall S \cdot S = V \setminus T \wedge S \subseteq n[S] \Rightarrow S = \varnothing$

- If $S = V \setminus T$, then
  $S = \varnothing \;\equiv\; V \setminus T = \varnothing \;\equiv\; V \subseteq T$
- The non-cycle condition then becomes
  $\forall S \cdot S = V \setminus T \wedge S \subseteq n[S] \Rightarrow V \subseteq T$



- Let us focus on $n[S]$
- Since $S = V \setminus T$, $n[S] = n[V \setminus T]$
- Since n is bijective, $n[V \setminus T]$ and $n[T]$ don't intersect (see figure on the right)
- Therefore, $n[V \setminus T] = n[V] \setminus n[T]$

From absence of cycles to induction

- Since $S = V \setminus T$ and $n[V \setminus T] = n[V] \setminus n[T]$, $S \subseteq n[S]$ becomes $V \setminus T \subseteq n[V] \setminus n[T]$
- Let us simplify that condition
- By definition: $f \in V$ and $f \notin n[V \setminus T]$ (f is not in the range of n)
- Since $V \setminus T \subseteq n[V \setminus T]$, $f \notin V \setminus T$
  (because $f \notin n[V \setminus T]$ and $V \setminus T$ contains a subset of $n[V \setminus T]$)
- Therefore f must be *subtracted* from V by T, and then $f \in T$
- Also by definition, $n[V] = V \setminus \{f\}$.
- So we can rewrite $V \setminus T \subseteq n[V] \setminus n[T]$ as $V \setminus T \subseteq (V \setminus \{f\}) \setminus n[T]$

From absence of cycles to induction

- Let us simplify $\underbrace{\overbrace{V}^{a} \setminus \overbrace{T}^{b}}_{e} \subseteq \underbrace{\overbrace{(V \setminus \{f\})}^{c} \setminus \overbrace{n[T]}^{d}}_{f}$.
- We know that $f \in V$ and $f \in T$.
- f is not in set (f), and then it should not be in (e); it is removed by (b).
- Then we have to worry about how much is removed by (b) and (d).
- If (d) removes “too much”, then (e) will be larger.
- I.e., if (d) contains an element that is not in (b), then (e) will contain an element that is not in (f).
- Therefore, (d) cannot contain elements that are not in (b).
- So the formula simplifies to $n[T] \subseteq T$.

From absence of cycles to induction

Putting all together, the non-cycle condition becomes

$$\forall S \cdot S = V \setminus T \land f \in T \land n[T] \subseteq T \Rightarrow V \subseteq T$$

If we expand $n[T] \subseteq T$:

$$\text{thm\_2} : \forall T \cdot f \in T \land (\forall x \cdot x \in T \Rightarrow n(x) \in T) \Rightarrow V \subseteq T$$

- T the set of elements with some property P: $T = \{x | P(x)\}$
- So the meaning of thm_2 is:
  - If the initial node f has property P ($f \in T$), and
  - For every element with property P ($x \in T$), the next one has this property ($n(x) \in T$), then
  - All elements have property P ($V \subseteq T$).

Using thm_2 to prove list properties

- We want to prove $P(x)$ for all $x \in V$.
- Elements for which P holds: $T = \{x | x \in V \wedge P(X)\}$.
- We want to prove that $T = V$.
- Since by construction $T \subseteq V$, it is enough to prove $V \subseteq T$.
- We do that by instantiating T in thm_2.

$$
\begin{array}{lr}
f \in \{x|x \in V \wedge P(x)\} & \wedge \\
\forall x \cdot x \in \{x|x \in V \wedge P(x)\} \Rightarrow n(x) \in \{x|x \in V \wedge P(x)\} & \Rightarrow \\
V \subseteq \{x|x \in V \wedge P(x)\} &
\end{array}
$$

- $f \in \{x|x \in V \wedge P(x)\} \equiv P(f)$.
- Second part equivalent to $\forall x \cdot x \in V \wedge P(x) \Rightarrow P(n(x))$.
- The RHS is equivalent to $\forall x \cdot x \in V \Rightarrow P(x)$.

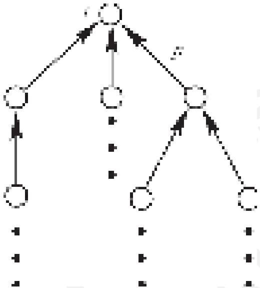
- Instantiating thm_2 gives a scheme to prove by induction in infinite lists.

Finite lists

- Basically as infinite lists, but including a last (l) element and a different axiom 2:

$$
\begin{aligned}
\text{axm\_4} : &\quad l \in V \\
\text{axm\_5} : &\quad \text{finite}(V) \\
\text{axm\_2}' : &\quad n \in V \setminus \{l\} \rightarrowtail\!\!\!\!\twoheadrightarrow V \setminus \{f\} \\
\text{induction} : &\quad \forall T \cdot T \subseteq V \land f \in T \land (\forall x \cdot x \in V \setminus \{l\} \land x \in T \Rightarrow n(x) \in T) \Rightarrow V \subseteq T
\end{aligned}
$$

Infinite trees



- t is the root.
- p links node with parent (surjection).
- No cycles.

axm_1 : $t \in V$

axm_2 : $p \in V \setminus \{t\} \twoheadrightarrow V$

axm_3 : $\forall S \cdot S \subseteq p^{-1}[S] \Rightarrow S = \varnothing$

Induction rule:

$$\forall T \cdot t \in T \wedge p^{-1}[T] \subseteq T \Rightarrow V \subseteq T$$

Instantiation to prove properties:

$$\begin{aligned} \forall T \cdot \quad & T \subseteq V \wedge t \in T \wedge \\ & (\forall x \cdot x \in V \setminus \{t\} \wedge p(x) \in T \Rightarrow x \in T) \\ & \Rightarrow V \subseteq T \end{aligned}$$

Note: placement of p in implication is *opposite* w.r.r. f for lists – “direction” of arrows reversed!

Finite trees

- t is the root.
- p relates every node with its parent.
- L is the set of tree leaves.
- There should not be cycles.

$$
\begin{aligned}
\text{axm\_1}: &\quad t \in V \\
\text{axm\_2}: &\quad L \subseteq V \\
\text{axm\_3}: &\quad p \in V \setminus \{t\} \twoheadrightarrow V \setminus L \\
\text{axm\_4}: &\quad \forall S \cdot S \subseteq p^{-1}[S] \Rightarrow S = \varnothing
\end{aligned}
$$

The induction scheme is as in infinite trees.