# Synchronizing Processes on a Tree Network[1]

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[1] Example and most slides borrowed from J. R. Abrial: see
http://wiki.event-b.org/index.php/Event-B_Language



## Purpose of this lecture

- Learning a few more modeling conventions.
- Learning more about abstraction.
- Formalizing and proving on an interesting structure: a tree.
  - Will have an intermediate step to review functions, relations, data structures.
- Study a more complicated problem in distributed computing
- Example studied in: *W.H.J. Feijen and A.J.M. van Gasteren. On a Method of Multi-programming. Springer Verlag, 1999.*

As usual:

- Define the informal requirements
- Define the refinement strategy
- Construct the various more and more concrete models

## Prerequisites

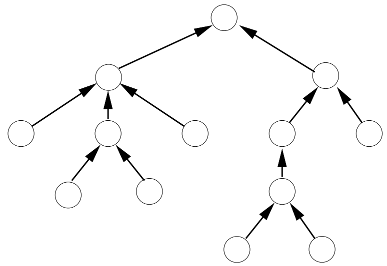
- Knowledge of first order logic, set theory, relations, and functions.
- Rodin (to discharge the proofs).
- Slides:
  - *Event B: Sets, Relations, Functions, Data Structures*
- Please go through them.
- I will review parts of it here, when needed.

## Comparison with previous examples

- Not a transformational system.
  - Not supposed to finish.
  - No final result.
- Not reactive.
  - No *external* world that reacts to system changes.
- Distributed.
  - Different *nodes* act autonomously.
  - With limited information access.
  - However, communication assumed to be reliable.
- Internal concurrency.
  - Every node has concurrent processes.
- Model small: just three events in the last refinement.
- However, proofs and reasoning involved.

## Requirements

| | |
|---|---|
| ENV 1 | We have a fixed set of processes forming a tree |



- Note: they do not need to form a tree from the beginning.
- A set of communicating processes can coordinate to form a tree.

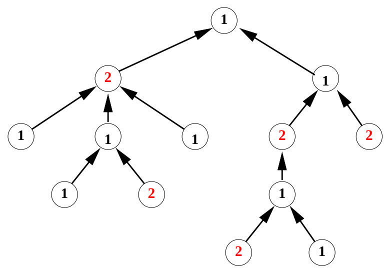
## Requirements (Cont.)

- All processes are supposed to execute forever the same code.
- But processes must remain (somewhat) synchronized.
- For this, each process has (initially) one counter.

| | |
|---|---|
| ENV 2 | Each process has a counter, which is a natural number |

- A process counter represents its "phase"
  (related to the work for which they have to synchronize).
- Difference between any two counters $\leq$ one.
- Each process is thus at most one phase ahead of the others

## Requirements (Cont.)




| | |
|---|---|
| FUN 3 | The difference between any two counters is at most equal to 1 |

## Requirements (Cont.)

- Reading the counters

| | |
|---|---|
| FUN 4 | Each process can read the counters of its immediate neighbors only |

- Modifying the counters

| | |
|---|---|
| FUN 5 | The counter of a process can be modified by this process only |

## Refinement strategy

- Construct abstract initial model dealing with FUN 3 and FUN 5
- Improve design to (partially) take care of FUN 4
- Improve design to better take care of FUN 4
- (Simplify final design to obtain efficient implementation).

| | |
|---|---|
| FUN 3 | The difference between any two counters is at most one |

| | |
|---|---|
| FUN 4 | Processes read counters of immediate neighbors only |

| | |
|---|---|
| FUN 5 | A process can modify only its counter(s) |

## Steps

1. Initial model: all nodes access to the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

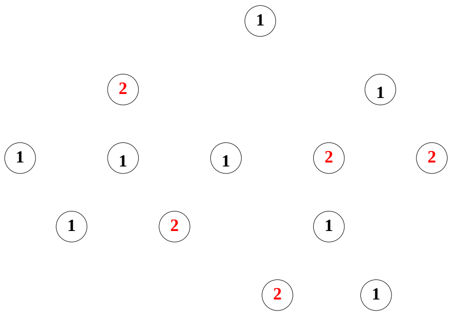
## Initial model: the state

- Simplify situation: forget about tree
- We just define the counters and express the main property: FUN 3

| | |
|---|---|
| FUN 3 | The difference between any two counters is at most one |

- The initial model is always far more abstract than the final system
- Other requirements are probably not fulfilled

## Abstract situation




| | |
|---|---|
| FUN 3 | The difference between any two counters is at most 1 |

Suggest an initial model!

## Initial model: the state

**carrier set:** $P$

**axm0_1:** $\mathrm{finite}(P)$

**variable:** $c$

**inv0_1:** $c \in P \to \mathbb{N}$

**inv0_2:** $\forall x, y \cdot \left( \begin{array}{l} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{array} \right)$

✓ *Create project `synch_tree`*
✓ *Create context `c0` with set, axiom*
✓ *Create machine `m0` with variable, invariants.*

## Is that right?

- inv0_2 may be surprising at first glance:

$$\forall x, y \cdot x \in P \land y \in P \Rightarrow c(x) \leq c(y) + 1$$

- Is it the same as $\forall i, j \cdot |c(i) - c(j)| \leq 1$?
- Disprove it or convince us!

## Is that right?

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- Is it the same as $\forall i, j \cdot |c(i) - c(j)| \leq 1$?
- Disprove it or convince us!

**Proof by double implication.**

Let us choose two arbitrary nodes with counters $a$ and $b$.

- If the invariant holds, then $a \leq b + 1$ and $b \leq a + 1$. From there, $a - b \leq 1$ and $b - a \leq 1$, therefore $|a - b| \leq 1$.
- If $|a - b| \leq 1$, then both $a - b \leq 1$ and $b - a \leq 1$. Then, inv0_2 is implied by the intended invariant.

## Initial model: events

```
init
  c := P × {0}
```

```
ascending
  any n where
    n ∈ P
    ∀m · m ∈ P ⇒ c(n) ≤ c(m)
  then
    c(n) := c(n) + 1
  end
```

- Note any $n$: it is logically $\exists n \cdot n \in P \land \cdots$
- Process counter incremented only when $\leq$ to all other counters.
- Intuition: *If I see I can increase without breaking difference constraint, I do it!*
- Non-determinism!
- A specification of what should happen.
- Not a final state (there is not one): a procedure that (hopefully) respects the invariant.

✓ *Add initialization, event*

Note: $\times$ is entered with **, any with pull-down menu, "Add event parameter".

## Proof of invariant preservation

$$
\begin{array}{ll}
c \in P \to \mathbb{N} & \textbf{inv0\_1} \\
\forall x, y \cdot \left( \begin{array}{l} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{array} \right) & \textbf{inv0\_2} \\
n \in P & \text{Guards of event} \\
\forall m \cdot (m \in P \Rightarrow c(n) \leq c(m)) & \text{ascending} \\
\vdash & \\
\forall x, y \cdot \left( \begin{array}{l} x \in P \\ y \in P \\ \Rightarrow \\ (c \Leftarrow \{n \mapsto c(n)+1\})(x) \leq (c \Leftarrow \{n \mapsto c(n)+1\})(y) + 1 \end{array} \right) &
\end{array}
$$

$$\Uparrow$$

Modified invariant **inv0_2**

In Rodin: automatic; if not, repeatedly apply lassoing, p0 or m0.

## Model so far

```
CONTEXT c0
SETS
    P
AXIOMS
    axm1: finite(P)
END
```

```
MACHINE m0
SEES c0
VARIABLES
    c
INVARIANTS
    inv1: c ∈ P → ℕ
    inv2: ∀x,y·x ∈ P ∧ y ∈ P ⇒ c(x) ≤ 1 + c(y)
EVENTS
Initialisation
    begin
        act1: c := P × {0}
    end
Event ascending ⟨ordinary⟩ ≙
    any
        n
    where
        grd12: n ∈ P
        grd11: ∀m·m ∈ P ⇒ c(n) ≤ c(m)
    then
        act11: c(n) := c(n) + 1
    end
```

## Problem with the current event

ascending
  **any** $n$ **where**
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
  **then**
    $c(n) := c(n) + 1$
  **end**

What requirement is this event breaking?

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  **any** $n$ **where**
    $n \in P$
    $\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$
  **then**
    $c(n) := c(n) + 1$
  **end**

What requirement is this event breaking?

| | |
|---|---|
| FUN 2 | Each node can read the counters of its immediate neighbors only |

## Steps

1. Initial model: all nodes access to the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
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## First refinement: (partially) solving the problem

- Introduce a designated process $r$.
- We suppose that the counter of $r$ is always minimal

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

- Rationale:
  - We only synchronize with $r$ — not compliant, but communication restricted.
  - Helps ensure that difference between any two nodes $\leq$ one.
  - Because: if for any $m$ either $c(m) = c(r)$ or $c(m) = c(r) + 1$, then difference between any $m, n \leq 1$.
- Treat this property as a new (temporary) invariant.

✓ *Extend `c0` into `c1` (left pane, right click, "Extend"), add constant $r$, axiom $r \in P$*
✓ *Refine `m0` into `m1` (left pane, right click, "Refine"), add new invariant*
✓ *`m0` should "see" `c1`*

## First refinement: proposal for the event refinement

We simplify the guard

| | |
|---|---|
| (abstract-)ascending<br>**any** $n$ **where**<br>$n \in P$<br>$\forall m \cdot m \in P \Rightarrow c(n) \leq c(m)$<br>**then**<br>$c(n) := c(n) + 1$<br>**end** | (concrete-)ascending<br>**any** $n$ **where**<br>$n \in P$<br>$c(n) = c(r)$<br>**then**<br>$c(n) := c(n) + 1$<br>**end** |

- Note: if $c(r)$ minimal, $c(n) < c(r)$ impossible; therefore $c(n) = c(r)$

  ✓ *Change "extended" to "not extended", change guard*
- We have then to prove guard strengthening.

## Guard strengthening

$c \in P \to \mathbb{N}$ **inv0_1**

$\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ c(x) \leq c(y) + 1 \end{pmatrix}$ **inv0_2**

$\forall m \cdot (m \in P \Rightarrow \boxed{c(r)} \leq c(m))$ **new invariant**

$n \in P$

$\boxed{c(n) = c(r)}$

Guards of concrete event ascending

$\vdash$

$n \in P$

$\forall m \cdot (m \in P \Rightarrow \boxed{c(n)} \leq c(m))$

Guards of abstract event ascending

In Rodin: lasso + p0

✓ *Go to the proving perspective, discharge proof*

## Model so far

`inv1` not discharged.

```
CONTEXT c1
EXTENDS c0
CONSTANTS
    r
AXIOMS
    axm1: r ∈ P
END
```

```
MACHINE m1
REFINES m0
SEES c1
VARIABLES
    c
INVARIANTS
    inv1: ∀m·m ∈ P ⇒ c(r) ≤ c(m)
EVENTS
Initialisation ⟨extended⟩
    begin
        act1: c := P × {0}
    end
Event ascending ⟨ordinary⟩ ≙
refines ascending
    any
        n
    where
        grd1: n ∈ P
        grd2: c(r) = c(n)
    then
        act1: c(n) := c(n) + 1
    end
END
```

## Pending problems

ascending
**any** $n$ **where**
$n \in P$
$c(n) = c(r)$
**then**
$c(n) := c(n) + 1$
**end**

$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$

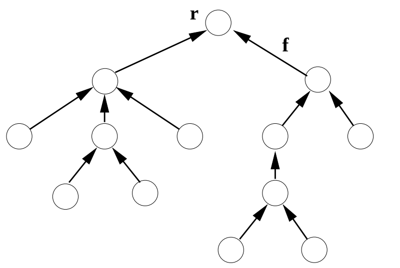
1. Prove that new "invariant" is preserved by the event.
2. The guard of the event still does not fulfill requirement FUN 4.

| | |
|---|---|
| FUN 4 | Each node can read the counters of its immediate neighbors only |

- Problem 1 solved in this refinement
- Problem 2 solved later

## First refinement: defining the tree

- Tree: root $r$ and "pointer" $f$ from each node in $P \setminus \{r\}$ to every node's parent.
- Plus some additional properties and inference rules.
- Reminder: sets, relations, functions, specific data structures and their inference rules.
- Note: constructing a tree ($\equiv$ root / leader + links) is a classical problem in distributed systems.
- Can also be tackled using Event B.




Invariant: we have a condition involving nodes in pairs and we need a condition that relates a single node $r$ with all the others.

## Update model

✓ *Add to* `c1` *(note $f$ is $\twoheadrightarrow$, written* `-»`*)*

- Constant $f$.
- Axioms:

$$L \subseteq P$$
$$f \in P \setminus \{r\} \twoheadrightarrow P \setminus L$$
$$\forall S \cdot S \subseteq f^{-1}[S] \Rightarrow S = \varnothing$$

- $f^{-1}$ is written f~.
- $\twoheadrightarrow$: $f$ defined for all $P \setminus \{r\}$ and *arrives* to every element in $P \setminus L$.

## Minimal counter at the root

- Minimality of counter at the root

$$\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$$

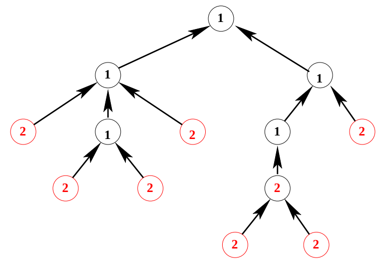
  relates $c(r)$ with $c(m)$ for every $m$.
- Events change local values and consult neighbouring values.
- We can (easily) prove properties relating neighbouring nodes.
- How can we relate local properties with global properties?

## Minimal counter at the root

- We define a weaker, local invariant first.
- The counter at every node is not greater than its descendants:

$$\text{inv1\_1}: \forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$$

✓ *Add it to* `m1`



**Rationale (advancing the algorithm)**

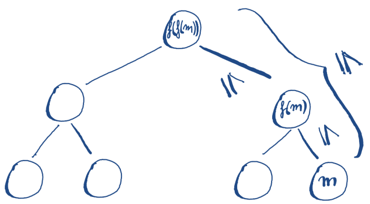
- Assume we can update the tree keeping a maximum difference between neighbors.
- Will be helpful to prove $c(r) \leq c(m)$.

## Minimal counter at the root: induction

We need to *extend* the local property

$\forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$

to the whole tree.




- Start with leaves $l \in L$.
- Prove that for any $l$, $c(f(l)) \leq c(l)$, then $c(f(f(l))) \leq c(f(l)) \leq c(l), \ldots$
- Work upwards towards root $r$.

**OR**

- Start with $r$.
- Prove that for all $m$ s.t. $r = f(m)$, $c(r) \leq c(m)$.
  $m$ is a child of $r$
- Then, for all $m'$ s.t. $m = f(m')$, $c(m) \leq c(m')\ldots$
- And so on towards the leaves.

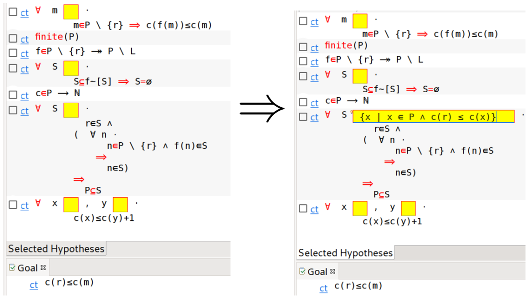
## Minimal counter at the root: induction

- Induction: difficult for theorem provers to do on their own.
  - Needs to identify base case, property to use for induction — i.e., the *strategy*.
- Proving property for base case & inductive step within theorem provers' capabilities.
- In Rodin: needs adding induction scheme:
  ✓ *Add to* `c1`*:*
  $\forall S \cdot S \subseteq P \land r \in S \land (\forall n \cdot n \in P \setminus \{r\} \land f(n) \in S \Rightarrow n \in S) \Rightarrow P \subseteq S$
  ✓ *Tip:* `Ctrl-Enter` *breaks text in input box in separate lines.*
- Instantiating it with the property to prove expressed as a set: `{x | x ∈ P ∧ c(r) ≤ c(x)}` (next slide)

✓ *In* `m1`*: ensure you have* inv1_1: $\forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$
✓ *Ensure* thm1_1: $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$ *below invariant, marked as theorem*

## Induction in Rodin: instantiation

- Double click in the unproved theorem (left pane).
- Switch to prover view, lasso.
- Locate induction axiom.
- Enter
  `{x | x ∈ P ∧ c(r) ≤ c(x)}.`
- Return and p0.
- The theorem should be proved now.




Invariant inv1_1 not yet proved. Requires order between parent and children $c(f(m)) \leq c(m)$ that ascending cannot guarantee: guard $c(r) = c(n)$ allows updates in arbitrary order. Will enforce through more local comparison.

## More local comparison

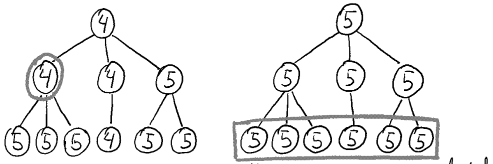
- Nodes with difference $\leq$ one from $r$.
- When can we update looking locally?

```
ascending
  any n where
    n ∈ P
    c(r) = c(n)
    ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
    c(n) := c(n) + 1
  end
```

Ensure inv1_1 is preserved: double click, prover view, lasso, p0 should do it.

$c(n) = c(r) \land \forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(m) > c(n)$

"There is a node that has not been updated and whose children have been updated"

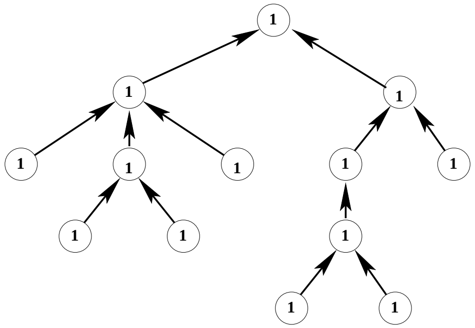


All of them can be updated

## How it is expected to work

Update order restricted:

- **Before:** any node whose counter is equal to the root (the one with the minimum).
- **Now:** only those nodes whose counters are, in addition, smaller than all its descendants.
- Updates will go in waves towards the root.

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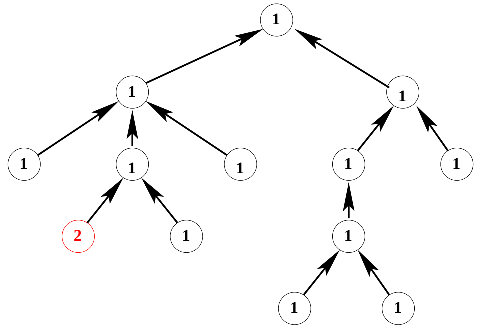


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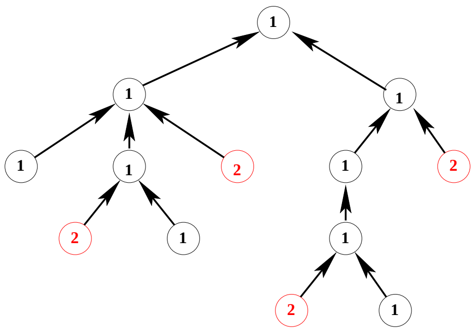


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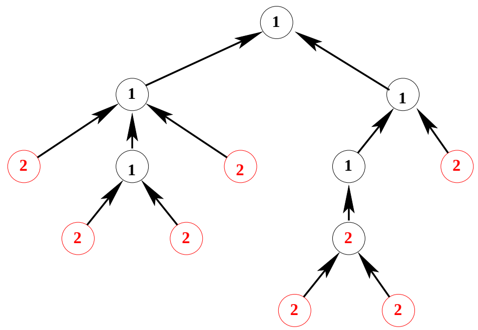

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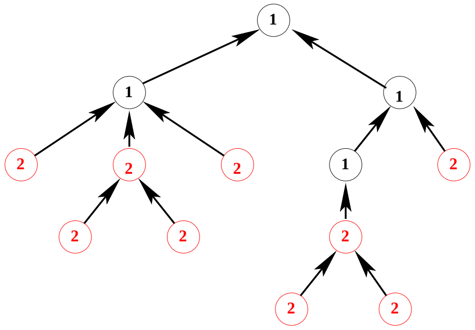

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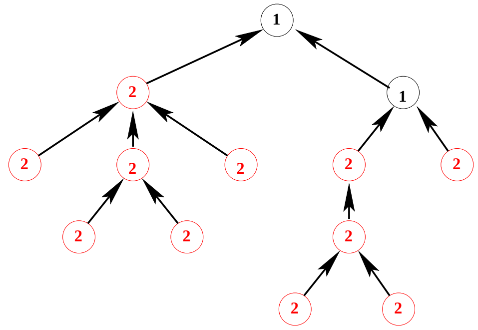

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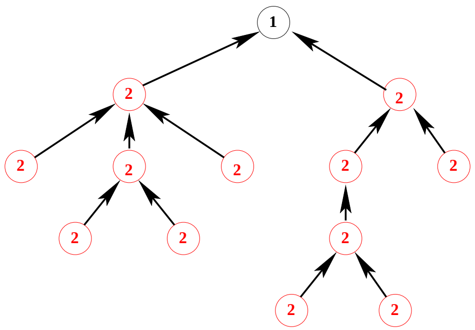

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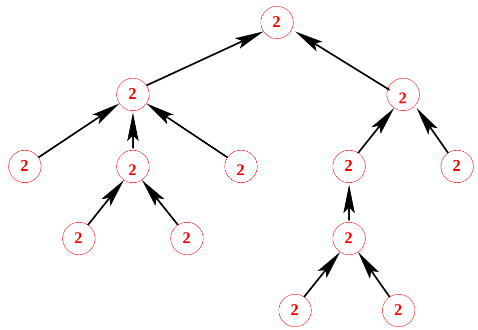

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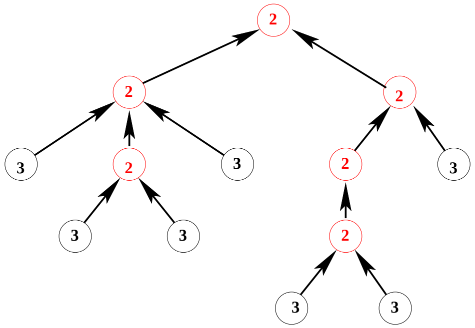


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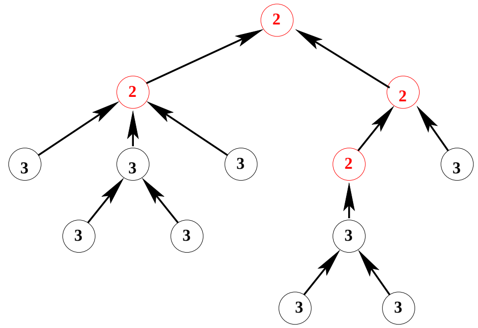


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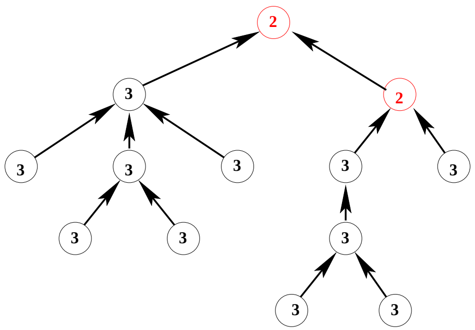


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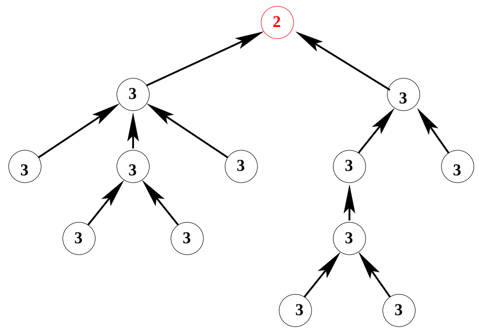


## Neighborhood checking

| FUN 4 | Each process can read the counters of its immediate neighbors only |
|---|---|

- $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) \neq c(m)$ uses only local comparisons.
- $c(r) = c(n)$ uses non-local comparisons.
- We will tackle that in the next refinement.

## Model so far

Note: $c(n) < c(m)$ in ascending should be $c(n) \neq c(m)$

**CONTEXT** c1
**EXTENDS** c0
**CONSTANTS**
  r
  f
  L
**AXIOMS**
  axm1: $r \in P$
  axm3: $L \subseteq P$
    Leaves
  axm2: $f \in P \setminus \{r\} \twoheadrightarrow P \setminus L$
  axm4: $\forall S \cdot S \subseteq f^{-1}[S] \Rightarrow S = \varnothing$
  axm5:
    $\forall S \cdot S \subseteq P \wedge$
    $r \in S \wedge$
    $(\forall n \cdot n \in P \setminus \{r\} \wedge f(n) \in S \Rightarrow n \in S)$
    $\Rightarrow$
    $P \subseteq S$
**END**

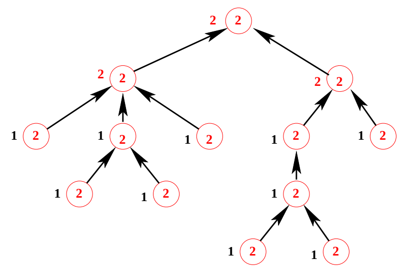
**MACHINE** m1
**REFINES** m0
**SEES** c1
**VARIABLES**
  c
**INVARIANTS**
  inv1: $\forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$
  inv2: ⟨theorem⟩ $\forall m \cdot m \in P \Rightarrow c(r) \leq c(m)$
**EVENTS**
**Initialisation** ⟨extended⟩
  **begin**
    act1: $c := P \times \{0\}$
  **end**
**Event** ascending ⟨ordinary⟩ $\widehat{=}$
**refines** ascending
  **any**
    n
  **where**
    grd1: $n \in P$
    grd2: $c(r) = c(n)$
    grd3: $\forall m \cdot m \in f^{-1}[\{n\}] \Rightarrow c(n) < c(m)$
  **then**
    act1: $c(n) := c(n) + 1$
  **end**
**END**

## Steps

1. Initial model: all nodes access to the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

## Second refinement

- Replace the guard $c(r) = c(n)$.
- Processes must be aware when this situation does occur.
- Add second counter $d(\cdot)$ to each node to capture value of $c(r)$.



## Second refinement: the state

| | |
|---|---|
| **carrier set:** | $P$ |
| **constants:** | $r, f$ |
| **variables:** | $c, d$ |

Invariant **inv2_2** is as **inv0_2**

**inv2_1:** $d \in P \to \mathbb{N}$

**inv2_2:** $\forall x, y \cdot \begin{pmatrix} x \in P \\ y \in P \\ \Rightarrow \\ d(x) \leq d(y) + 1 \end{pmatrix}$

$d$ has an overall property similar to $c$:

$\forall x, y \cdot x \in P \wedge y \in P \Rightarrow c(x) \leq c(y) + 1$

- $d$ will capture the value of $c(r)$.
- It will be updated in a downward wave.

✓ *Refine* `m1` *into* `m2`

✓ *Add variable* $d$ *and invariants*

## Updating $d$

This refinement captures:

- The existence of $d$.
- How its update can proceed not to break its invariant.

```
Event descending
    any n where
        n ∈ P
        ∀m · m ∈ P ⇒ d(n) ≤ d(m)
    then
        d(n) := d(n) + 1
    end
```

✓ *Add event to* `m2`

✓ *Initialize* $d$ *to 0 (copy the initialization of* $c$*)*

## Steps

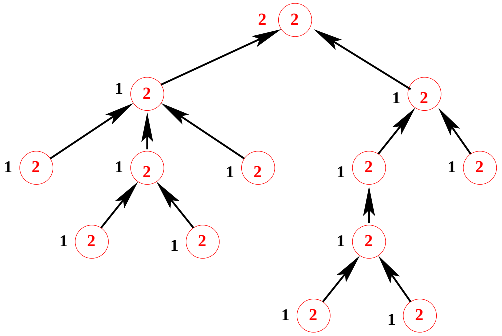
1. Initial model: all nodes access to the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

## Third refinement

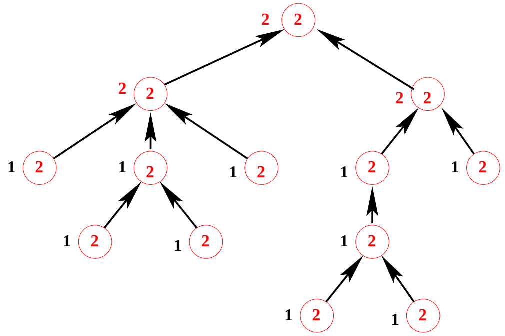
- We extend the invariant of counter $d$.
- We establish the relationship between both counters $c$ and $d$.
  - This will allow us to refine event ascending
- We construct the descending wave (by refining event descending).
- Remark: this is the most difficult refinement.

✓ *Refine* `m2` *into* `m3`

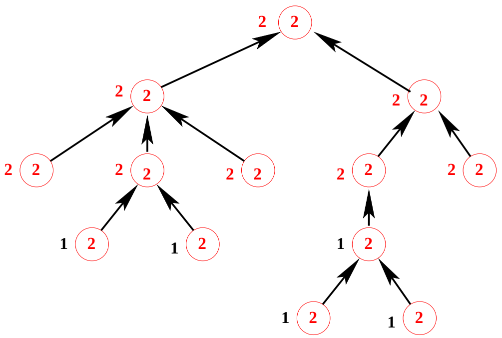
## Idea behind third refinement



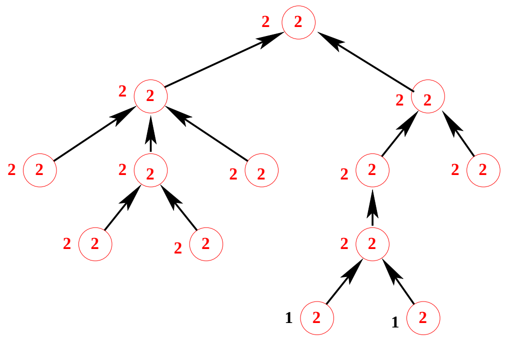

## Idea behind third refinement



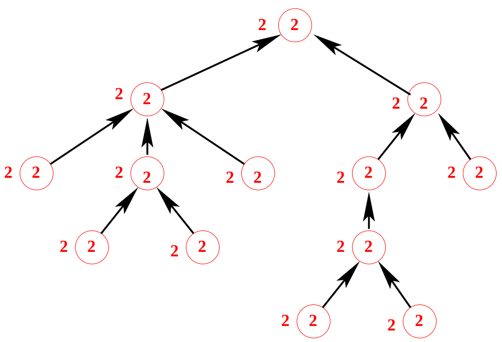

## Idea behind third refinement



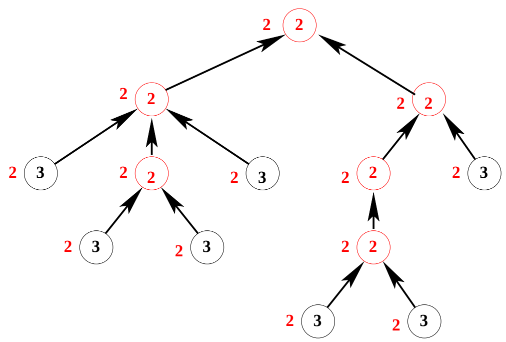

## Idea behind third refinement



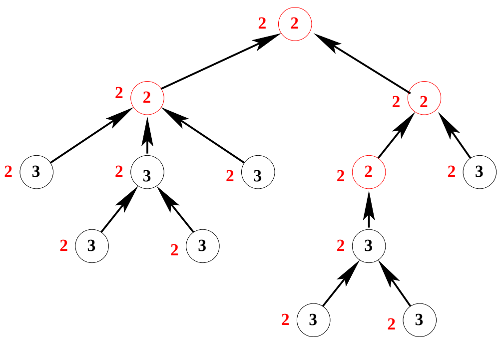
## Idea behind third refinement



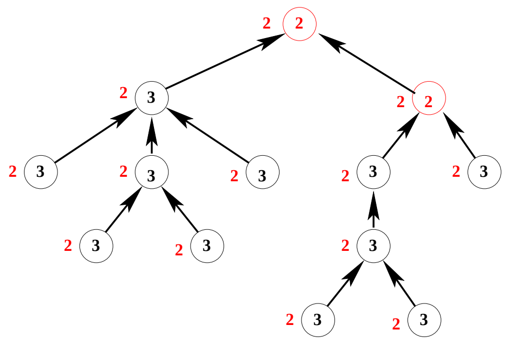
## Idea behind third refinement



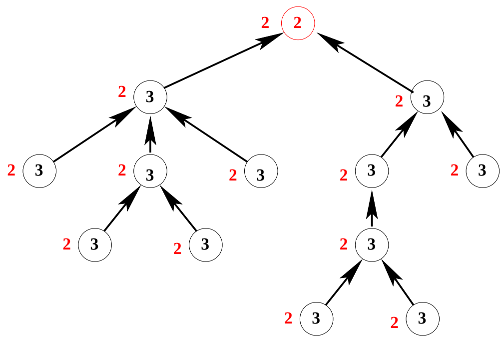
## Idea behind third refinement



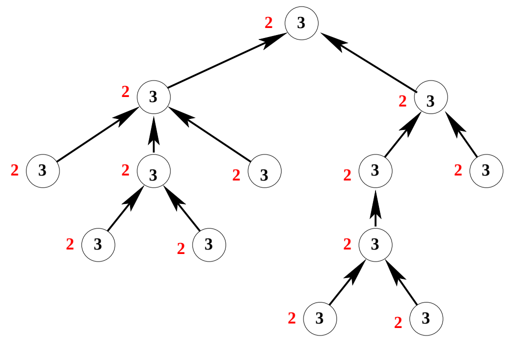
## Idea behind third refinement



## Idea behind third refinement




## Idea behind third refinement




## State and invariants

- Recall local condition for $c$:

$$\text{inv1\_1} : \forall m \cdot m \in P \setminus \{r\} \Rightarrow c(f(m)) \leq c(m)$$

  *Every node's counter is smaller than or equal to its children's.*

- Local condition for $d$ is similar:

$$\text{inv3\_1} : \forall m \cdot m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m))$$

  *Every node's counter is smaller than or equal to its parent (if it has a parent).* This is what makes the wave *descending*.

- inv3_1 and tree induction proves that the root has the highest value of $d(\cdot)$:

$$\text{thm3\_1} : \forall n \cdot n \in P \Rightarrow d(n) \leq d(r)$$

  *(remember: root had the smallest value of* $c(\cdot)$)

## Proving theorem and invariant

- ✓ *Add to* `m3`*:*

$$\textit{inv3\_1} : \quad \forall m \cdot m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m))$$
$$\textit{thm3\_1} : \quad \forall n \cdot n \in P \Rightarrow d(n) \leq d(r)$$

- ✓ *Mark the latter as theorem*
- ✓ *Double click on the PO for THM*
- ✓ *Go to proving perspective; locate induction axiom*
- ✓ *Instantiate with* $\{x | x \in P \land d(x) \leq d(r)\}$*, invoke p0*
- ✓ *That should prove* `thm3_1`
- ✓ *inv3_1 cannot be proved yet - reasons similar to* `c`*.*
  *We will deal with that later*

## Refining *ascending*

```
Event (abstract-)ascending
  any n where
    n ∈ P
    c(n) = c(r)
    ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
    c(n) := c(n) + 1
  end
```

```
Event (concrete-)ascending
  any n where
    n ∈ P
    c(n) = d(n)
    ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
    c(n) := c(n) + 1
  end
```

- Downward wave $d$ will eventually propagate $d(r)$.
  ✓ *Change event guard in* `m3`

ascending: only local comparisons now!

## Refining *ascending*

```
Event (abstract-)ascending
  any n where
    n ∈ P
    c(n) = c(r)
    ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
    c(n) := c(n) + 1
  end
```

```
Event (concrete-)ascending
  any n where
    n ∈ P
    c(n) = d(n)
    ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
    c(n) := c(n) + 1
  end
```

- Downward wave $d$ will eventually propagate $d(r)$.
  ✓ *Change event guard in* `m3`
- Need to prove guard strengthening.

ascending: only local comparisons now!

## Refining *ascending*

```
Event (abstract-)ascending
  any n where
    n ∈ P
    c(n) = c(r)
    ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
    c(n) := c(n) + 1
  end
```

```
Event (concrete-)ascending
  any n where
    n ∈ P
    c(n) = d(n)
    ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
    c(n) := c(n) + 1
  end
```

- Downward wave $d$ will eventually propagate $d(r)$.
  ✓ *Change event guard in* `m3`
- Need to prove guard strengthening.
- We cannot. $c$ and $d$ unrelated so far!
  ✓ *Relate* $c$ *and* $d$*:* inv3_2 : $d(r) \leq c(r)$
- If needed: proving perspective, lasso + p0 proves strengthening.

ascending: only local comparisons now!

## Refining *descending*

- A different case.
- Two situations raise a change of $d$:
  1. For a non-root node: parent's $d$ change.
  2. For the root node: $c(r)$ changes.
- Different guards.
- We will prepare the events to be edited.

✓ *Change (concrete) descending event to non-extended*
✓ *Left click on circle to left of name to select*
`Ctrl-C` *to copy,* `Ctrl-V` *to paste*
✓ *Rename first event as descending_nr.*
✓ *Rename second event as descending_r.*

## Refining *descending*: the non-root case

```
Event (abstract-)descending
    any n where
        n ∈ P
        ∀m · m ∈ N ⇒ d(n) ≤ d(m)
    then
        d(n) := d(n) + 1
    end
```

```
Event (concrete-)descending
    any n where
        n ∈ P\{r}
        d(n) ≠ d(f(n))
    then
        d(n) := d(n) + 1
    end
```

✓ *Update guards*

(Note: Rodin $\geq$ 3.6 seems to prove strengthening automatically; previous versions needed additional steps [in next slide])

## Proving guard strengthening

Note: the steps below do not seem to be necessary in Rodin 3.6 with the Atelier B provers installed. Strengthening is proven automatically.

$$n \in P\backslash\{r\}, d(n) = d(f(n)), m \in P \vdash d(n) \leq d(m)$$

We need some magic mushrooms to help the provers:

thm3_2 : $\forall n \cdot n \in P\backslash\{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1$

thm3_3 : $\forall n \cdot n \in P \Rightarrow d(r) \in d(n)..d(n)+1$

thm3_2 downward wave, parent is at most one more than children (when it has just been increased)

thm3_3 special case for root (the first one to be increased)

## Refining *descending* (Cont. — the root case.)

```
Event (abstract-)descending
    any n where
        n ∈ P
        ∀m · m ∈ P ⇒ d(n) ≤ d(m)
    then
        d(n) := d(n) + 1
    end
```

```
Event (concrete-)descending
    refines
        descending
    when
        d(r) ≠ c(r)
    with
        n: n = r
    then
        d(r) := d(r) + 1
    end
```

✓ *Click on circle left of param. $n$, delete*

- Parameter $n$ disappeared!
- Substitute *(witness)* for GRD, SIM.
- We are particularizing for $r$.
- Similar to gluing invariant!

- Note with label: specific Rodin idiom.
- Need to prove $d(r) \neq c(r),\ m \in P \vdash d(r) \leq d(m)$
- ML should do the trick.

## Finishing proofs

**Note: this is the version I had in previous courses. It seems that with Rodin 3.6, ML as applied in the previous slide does the trick. Or that I did not bother to try it...**

I needed two more magic pills:

inv3_3 : $\forall n \cdot n \in P \Rightarrow c(n) \in d(n)..d(n)+1$ To prove GRD

thm3_4 : $\forall n \cdot n \in P \Rightarrow c(r) \in d(n)..d(n)+1$ To prove inv3_3

Plus, if not added before:

thm3_2 : $\forall n \cdot n \in P\backslash\{r\} \Rightarrow d(f(n)) \in d(n)..d(n)+1$

thm3_3 : $\forall n \cdot n \in P \Rightarrow d(r) \in d(n)..d(n)+1$

After this, the invariant can be proved with a combination of several steps:

- Apply lasso.
- Instantiate $\forall n \cdot c(r) \in d(n)..d(n)+1$ (which relates $c$ and $d$) with $n$.
- Remove $\in$ in goal ($c(n) \in d(n)+1..d(n)+1+1$) to create inequalities.
- Do P0 in $c(n) \leq d(n)+1+1$ goal.
- Note that only possibility to prove is $d(n) = c(n)$.
- Do case distinction with $d(n) = c(n)$,
- Apply ML to the subgoals.

## Finishing proofs

**Note: this strategy works with Rodin 3.6**

We needed one magic pill:

$$\text{inv3\_3}: \quad \forall n \cdot n \in P \Rightarrow c(n) \in d(n)..d(n)+1 \quad \text{To prove GRD}$$

After this, the invariant can be proved with a combination of several steps:

- Apply lasso.
- Apply ML to $c(n0) \in d(n0)..d(n0)+1$.
-
- Remove $\in$ in goal ($c(n) \in d(n)+1..d(n)+1+1$) to create inequalities.
- Do ML in $c(n) \leq d(n)+1+1$ goal.
- For $d(n)+1 \leq c(n)$, do case distinction:
  - Either with $d(n) = c(n)$, or
  - with $d(n)+1 = c(n)$

  and ML to the subgoals.

## Third refinement: invariants

**inv3_1:** $\forall m \cdot (m \in P \setminus \{r\} \Rightarrow d(m) \leq d(f(m)))$

**inv3_2:** $d(r) \leq c(r)$

**inv3_3:** $\forall n \cdot (n \in P \Rightarrow c(n) \in d(n)\,..\,d(n)+1)$

**thm3_1:** $\forall m \cdot (m \in P \Rightarrow d(m) \leq d(r))$

**thm3_2:** $\forall n \cdot (n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)\,..\,d(n)+1)$

**thm3_3:** $\forall n \cdot (n \in P \Rightarrow d(r) \in d(n)\,..\,d(n)+1)$

**thm3_4:** $\forall n \cdot (n \in P \Rightarrow c(r) \in d(n)\,..\,d(n)+1)$

## Third refinement: events

```
Event descending_r
  when
      d(r) ≠ c(r)
  with
    n: n = r
  then
      d(r) := d(r) + 1
  end
```

```
Event descending_nr
  any n where
      n ∈ P\{r}
      d(n) ≠ d(f(n))
  then
      d(n) := d(n) + 1
  end
```

```
Event ascending
  any n where
      n ∈ P
      c(n) = d(n)
      ∀m · m ∈ f⁻¹[{n}] ⇒ c(n) ≠ c(m)
  then
      c(n) := c(n) + 1
  end
```

## Steps

1. Initial model: all nodes access to the state of all nodes.
2. First refinement: restrict access to a single node.
3. Second refinement: local check, upwards wave.
4. Third refinement: construct downwards wave.
5. Fourth refinement: remove upwards and downwards counters.

## Observation

- The difference among counters is at most one.
  - That has proven by construction.
- In the guards, we only care whether they are equal or not.
- For this, we only need parity!

$$a, b \in \mathbb{N} \land |a - b| \leq 1 \Rightarrow (a = b \Leftrightarrow parity(a) = parity(b))$$

- We will prove that this is a valid refinement.

✓ *Extend context c1 into c2*
✓ *Refine m3 into m4*
✓ *m4 should see c2*

## Formalizing parity

- We replace the counters by their parities
- we add the constant $parity$

**carrier set:** $P$

**constants:** $r, f, parity$

**axm4_1:** $parity \in \mathbb{N} \to \{0, 1\}$

**axm4_2:** $parity(0) = 0$

**axm4_2:** $\forall x \,.\, (x \in \mathbb{N} \Rightarrow parity(x + 1) = 1 - parity(x))$

✓ *Add parity and axioms to c2. Note: parity is a function!*
✓ *Need some clicking (dom to $\mathbb{N}$ + ML ) to prove WD*

## The definitions that replace $c(\cdot)$ and $d(\cdot)$

- We replace $c$ and $d$ by $p$ and $q$

**variables:** $p, q$

**inv4_1:** $p \in P \to \{0, 1\}$

**inv4_2:** $q \in P \to \{0, 1\}$

**inv4_3:** $\forall n \,.\, (n \in P \Rightarrow p(n) = parity(c(n)))$

**inv4_4:** $\forall n \,.\, (n \in P \Rightarrow q(n) = parity(d(n)))$

✓ *Do it in m4. Note the gluing invariants! p and q really syntactic sugar.*

## New events: counters replaced by parity

ascending
  **any** $n$ **where**
    $n \in P$
    $p(n) = q(n)$
    $\forall m \cdot (m \in f^{-1}[\{n\}] \Rightarrow p(m) \neq p(n))$
  **then**
    $p(n) := 1 - p(n)$
  **end**

descending_1
  **any** $n$ **where**
    $n \in P \setminus \{r\}$
    $q(n) \neq q(f(n))$
  **then**
    $q(n) := 1 - q(n)$
  **end**

descending_2
  **when**
    $p(r) \neq q(r)$
  **then**
    $q(r) := 1 - q(r)$
  **end**

## Proving remaining POs (in `ascending`)

GRD of $q(n) = p(n)$

- Needs additional property

$$\forall x, y \cdot y \in \mathbb{N} \land x \in y..y+1 \quad \Rightarrow \quad (parity(x) = parity(y) \Leftrightarrow x = y)$$

- We could make it axiom, but it can be proven as theorem (better!).
- Proving it is not difficult.

WD: P0 takes care of it.
THM: A couple of simple rewritings + distinction by cases work.

- $\Longleftrightarrow$: rewrite in two implications.
- par($x$) = par($y$) $\Rightarrow x = y$: ah with possible values of $x$.
- Prove ah with ML.
- Goal $y = y + 1$: do dc with par$(y) = 0$.
- P0 works for both branches.

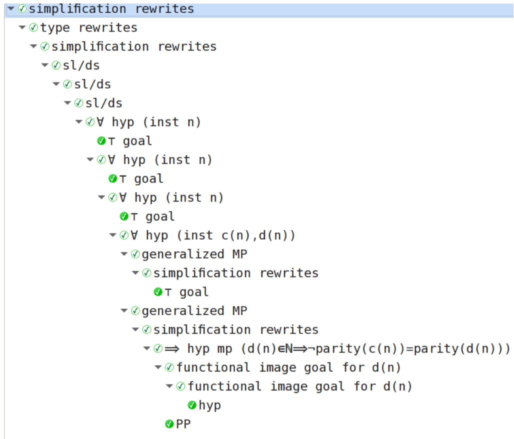
## Proving remaining POs (in `ascending`)

GRD of $q(n) = p(n)$

- With theorem

$$\forall x, y \cdot y \in \mathbb{N} \land x \in y..y+1 \quad \Rightarrow \quad (parity(x) = parity(y) \Leftrightarrow x = y)$$

- Instantiate with $c(n)$, $d(n)$.
- Instantiate defs. of $p(n)$, $q(n)$.
- Invoke P0.




## Proving POs (in `ascending`)

GRD of $\forall m \cdot m \in f^{\sim}[n] \Rightarrow p(n) \neq p(m)$

One simple path that works:

1. Add a new THM: $\forall n \cdot n \in P \setminus \{r\} \Rightarrow c(n) \in c(f(n))..c(f(n)) + 1$
2. Introduce the hypothesis $n = f(m)$ (which comes from $m \in f^{-1}[n]$) with ah and use ML repeatedly. See recording at course web.

Rationale: we have to prove than if $p(m) \neq p(f(m))$, then $c(n) \neq c(f(m))$. We have a theorem that says $parity(x) = parity(y) \Leftrightarrow x = y$ when $x \in y..y+1$. So we need $c(n) \in c(f(n))..c(f(n)) + 1$ to apply it. We add it as a theorem, which is immediately proven, and ML can use it.

## Discharging POs (in `descending`)

- In my case, GRD for $q(n) \neq q(f(n))$ in **descending_nr** remains to be proven.
- It should imply $d(n) \neq d(f(n))$.
- Similar to the previous case.
- Add a symmetrical theorem $\forall n \cdot n \in P \setminus \{r\} \Rightarrow d(f(n)) \in d(n)..d(n) + 1$
- It is immediately proven and it discharges the pending GRD proof.

## Less Manual Work?

- *Atelier B* provers: integrated and developed in conjunction with Rodin and with Event B in mind.
- However, in the world of theorem provers probably not the most powerful ones.
- Some third-party SMT provers available as plugins.
  - Check term project for installation instructions.
- Not guaranteed to work always seamlessly.
- But in many cases can discharge proofs without manual intervention!
- Why not using them before?
  - I wanted to show interactive theorem proving in examples that are not too complex to require it.