Event B: First-Order Logic, Sets, Relations, Functions, Arithmetic[1]

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[1] Many slides borrowed from J. R. Abrial: see `http://wiki.event-b.org/index.php/Event-B_Language`

The first-order predicate calculus and its rules

- Handling of variables, expressions, quantifiers, instantiation.
- An expression is a formal text denoting an object.
- A predicate denotes nothing.
- An expression cannot be proved.
- A predicate cannot be evaluated.
- Predicates and expressions are not interchangeable.
- Expressions will be extended with set-theoretic and arithmetic notation.

First-order predicate calculus: informal

We have a universe of objects. We make statements about these objects. *Sweet Reason* [HGTA11] is a delightful introduction to logic with examples.

$\forall x \cdot P(x)$: For all elements x, P holds. P can be arbitrarily complex.

$\exists x \cdot P(x)$: For some element x, P holds. P can be arbitrarily complex.

First-order predicate calculus: informal

$l(x, y)$	x loves y
$\forall x \cdot \forall y \cdot l(x, y)$	
$\exists x \cdot \exists y \cdot l(x, y)$	
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$\exists y \cdot \forall x \cdot l(x, y)$	
$\forall y \cdot \exists x \cdot l(x, y)$	
$\exists x \cdot \forall y \cdot l(x, y)$	
$\forall x \cdot \neg l(x, x)$	

We usually want to prove these statements true or false. We use inference rules to prove truth or falsehood.

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$\forall x \cdot \neg l(x, x)$	no one loves themself

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Some deductions and (non) equivalences

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(definition of existential quantifier)

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$$P(a) \Rightarrow \exists x \cdot P(x)$$

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$$P(a) \Rightarrow \exists x \cdot P(x)$$

$$\forall x \cdot (P(x) \Rightarrow B) \equiv (\exists x \cdot P(x) \Rightarrow B)$$
$$(x \notin vars(B))$$

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$$\forall x \cdot (P(x) \wedge Q(x)) \equiv \forall x \cdot P(x) \wedge \forall x \cdot Q(x)$$

$$\exists x \cdot (P(x) \vee Q(x)) \equiv \exists x \cdot P(x) \vee \exists x \cdot Q(x)$$

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$P(a) \Rightarrow \exists x \cdot P(x)$

$\forall x \cdot (P(x) \Rightarrow B) \equiv (\exists x \cdot P(x) \Rightarrow B)$
$(x \notin vars(B))$

$\forall x \cdot (P(x) \land Q(x)) \equiv \forall x \cdot P(x) \land \forall x \cdot Q(x)$

$\exists x \cdot (P(x) \lor Q(x)) \equiv \exists x \cdot P(x) \lor \exists x \cdot Q(x)$

$\forall x \cdot (P(x) \lor Q(x)) \not\equiv \forall x \cdot P(x) \lor \forall x \cdot Q(x)$

(Counterexample?)

$\exists x \cdot (P(x) \land Q(x)) \not\equiv \exists x \cdot P(x) \land \exists x \cdot Q(x)$

(Counterexample?)

First-order predicate calculus: inference rules

$$\frac{\mathbf{H},\ \forall \mathbf{x} \cdot \mathbf{P(x)},\ \mathbf{P(E)}\ \vdash\ \mathbf{Q}}{\mathbf{H},\ \forall \mathbf{x} \cdot \mathbf{P(x)}\ \vdash\ \mathbf{Q}}\ \textbf{ALL\_L}$$

where **E** is an expression

$$\frac{\mathbf{H}\ \vdash\ \mathbf{P(x)}}{\mathbf{H}\ \vdash\ \forall \mathbf{x} \cdot \mathbf{P(x)}}\ \textbf{ALL\_R}$$

- In rule **ALL_R**, variable **x** is not free in **H**

First-order predicate calculus: inference rules

$$\frac{\mathbf{H},\ \mathbf{P(x)}\ \vdash\ \mathbf{Q}}{\mathbf{H},\ \exists \mathbf{x} \cdot \mathbf{P(x)}\ \vdash\ \mathbf{Q}}\ \textbf{XST\_L}$$

- In rule **XST_L**, variable **x** is not free in **H** and **Q**

$$\frac{\mathbf{H}\ \vdash\ \mathbf{P(E)}}{\mathbf{H}\ \vdash\ \exists \mathbf{x} \cdot \mathbf{P(x)}}\ \textbf{XST\_R}$$

where **E** is an expression

First-order predicate calculus: inference rules

Compare rules:

$$\frac{\mathbf{H},\ \forall \mathbf{x} \cdot \mathbf{P(x)},\ \mathbf{P(E)}\ \vdash\ \mathbf{Q}}{\mathbf{H},\ \forall \mathbf{x} \cdot \mathbf{P(x)}\ \vdash\ \mathbf{Q}}\ \textbf{ALL\_L} \qquad \frac{\mathbf{H}\ \vdash\ \mathbf{P(x)}}{\mathbf{H}\ \vdash\ \forall \mathbf{x} \cdot \mathbf{P(x)}}\ \textbf{ALL\_R}$$

$$\frac{\mathbf{H},\ \mathbf{P(x)}\ \vdash\ \mathbf{Q}}{\mathbf{H},\ \exists \mathbf{x} \cdot \mathbf{P(x)}\ \vdash\ \mathbf{Q}}\ \textbf{XST\_L} \qquad \frac{\mathbf{H}\ \vdash\ \mathbf{P(E)}}{\mathbf{H}\ \vdash\ \exists \mathbf{x} \cdot \mathbf{P(x)}}\ \textbf{XST\_R}$$

First-order predicate calculus: inference rules

Rules for equality (some already seen):

$$\frac{\mathsf{H(F)},\ \mathsf{E} = \mathsf{F} \ \vdash\ \mathsf{P(F)}}{\mathsf{H(E)},\ \mathsf{E} = \mathsf{F} \ \vdash\ \mathsf{P(E)}} \quad \textbf{EQ\_LR} \qquad \frac{\mathsf{H(E)},\ \mathsf{E} = \mathsf{F} \ \vdash\ \mathsf{P(E)}}{\mathsf{H(F)},\ \mathsf{E} = \mathsf{F} \ \vdash\ \mathsf{P(F)}} \quad \textbf{EQ\_RL}$$

$$\frac{}{\vdash\ \mathsf{E} = \mathsf{E}} \quad \textbf{EQL}$$

$$\frac{\mathsf{H} \ \vdash\ \mathsf{E} = \mathsf{G} \ \wedge\ \mathsf{F} = \mathsf{I}}{\mathsf{H} \ \vdash\ \mathsf{E} \mapsto \mathsf{F} = \mathsf{G} \mapsto \mathsf{I}} \quad \textbf{PAIR}$$

Note: $\mathsf{E} \mapsto \mathsf{F}$ denotes a *pair* (E, F) — we will use them later.

Set theory: membership

- Event-B formal reasoning is built based on:
  - First-order logic inference rules (seen).
  - Set theory (to be touched upon).
- Set theory as a foundation for relations, functions (and, therefore, data structures).
  - Proofs often reduced to proving goals on sets.
- We will briefly see how this is intuitively done.

Set theory: membership

- A set is a well-defined collection of distinct objects.
- Set theory is primary concerned the membership predicate

$$\mathsf{E} \in \mathsf{S}$$

- E is an expression, S is a set.

Set theory: basic constructs

There are three basic constructs in set theory:

Cartesian product	$S \times T$
Power set	$\mathbb{P}(S)$
Comprehension 1	$\{ x \cdot x \in S \ \wedge\ P(x) \mid F(x) \}$
Comprehension 2	$\{ x \mid x \in S \ \wedge\ P(x) \}$

S and T are sets, x is a variable, P is a predicate, F is an expression.

Set theory: basic constructs
Definitions

Defined by equivalences

$$
\begin{array}{rcl}
E \mapsto F \in S \times T & \equiv & E \in S \wedge F \in T \\
S \in \mathbb{P}(T) & \equiv & \forall x \cdot x \in S \Rightarrow x \in T \\
E \in \{x \mid x \in S \wedge P(x)\} & \equiv & E \in S \wedge P(E) \\
E \in \{x \cdot x \in S \wedge P(x) \mid F(x)\} & \equiv & \exists x \cdot x \in S \wedge P(x) \wedge E = F(x)
\end{array}
$$

Set theory: basic constructs
Examples

$$
\begin{array}{rcl}
\{1,2,3\} \times \{a,b\} & = & \{1 \mapsto a, 1 \mapsto b, 2 \mapsto a, 2 \mapsto b, 3 \mapsto a, 3 \mapsto b\} \\
\mathbb{P}(\{1,2,3\}) & = & \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\varnothing\} \\
\{x \mid x \in \{2,3,4,5\} \wedge x \bmod 2 = 0\} & = & \{2,4\} \\
\{x \cdot x \in \{2,3,4,5\} \wedge x \bmod 2 = 1 \mid x^2\} & = & \{25,9\}
\end{array}
$$

Reminder: $A \mapsto B$ is a tuple.
It is sometimes written as (A, B) in other formalisms.

Shortcut: $m..n \equiv \{x \in \mathbb{Z} \mid m \leq x \wedge x \leq n\}$

- $\{x \mid x \in \mathbb{N} \wedge x < 2\} \times 8..10$
- $\{x \cdot x \in 3..5 \mid x \mapsto x * x\}$
- $\{n \cdot n \in \mathbb{N} \mid (0..n) \mapsto n\}$

Operations on sets

$$
\begin{array}{rcl}
S \subseteq T & \equiv & S \in \mathbb{P}(T) \\
S = T & \equiv & S \subseteq T \wedge T \subseteq S \\
S \cup T & \equiv & \{x \mid x \in S \vee x \in T\} \\
S \cap T & \equiv & \{x \mid x \in S \wedge x \in T\} \\
S \setminus T & \equiv & \{x \mid x \in S \wedge x \notin T\} \\
E \in \{a, \ldots, z\} & \equiv & E = a \vee \ldots \vee E = z \\
E \in \varnothing & \equiv & \perp
\end{array}
$$

- Operators based on membership and logic operations.
- Note: $E \notin T \equiv \neg(E \in T)$.
- Also: generalized / conditional union and intersection (see reference cards).

Binay relations

- A binary relation $r \in S \leftrightarrow T$ is a subset of their Cartesian product: $r \subseteq S \times T$
- Different syntax to highlight structure.
- $S \leftrightarrow T$: all (= the set of) the possible relations between S and T.
  - r would be one of them.
- $r \in 1..3 \leftrightarrow 7..11$
  - $r = \{1 \mapsto 10, 2 \mapsto 7, 2 \mapsto 11\}$
  - $4 \mapsto 10 \notin r$

$$
\begin{array}{rcl}
x \in dom(r) & \equiv & \exists y \cdot x \mapsto y \in r \\
y \in ran(r) & \equiv & \exists x \cdot x \mapsto y \in r \\
r^{-1} & \equiv & \{y \mapsto x \mid x \mapsto y \in r\}
\end{array}
$$

- $r \in \{\text{meat}, \text{fish}, \text{pasta}, \text{bacon}\} \leftrightarrow \{\text{carbs}, \text{protein}, \text{fat}\}$ – write a couple of relations.
- $dom(r)$, $ran(r)$, relation with S and T
- How many different r may there be?

Types of relations

Total	$S \leftrightarrow\!\!\!\!\!\leftarrow T$	$r \in S \leftrightarrow T \wedge dom(r) = S$
Surjective	$S \leftrightarrow\!\!\!\!\!\rightarrow T$	$r \in S \leftrightarrow T \wedge ran(r) = T$
Both	$S \twoheadleftarrow\!\!\!\!\!\twoheadrightarrow T$	$r \in S \leftrightarrow\!\!\!\!\!\leftarrow T \wedge r \in S \leftrightarrow\!\!\!\!\!\rightarrow T$

Can you classify the following relations?
(Use common sense; we are not looking for hidden corner cases)

- $Satellite \in SkyBodies \leftrightarrow SkyBodies$
- $Riding \in Person \leftrightarrow MovingBicycle$
- $BirthDate \in LivingPerson \leftrightarrow Date$
- $SquareRoot \in \mathbb{R} \leftrightarrow \mathbb{R}$
- $ThePreviousMoment \in Time \leftrightarrow Time$

Hint: sets and relations are very useful modeling tools!

Operations on relations

Domain restriction	$S \lhd r$	$\{x \mapsto y \in r \mid x \in S\}$
Domain subtraction	$S \ntriangleleft r$	$\{x \mapsto y \in r \mid x \notin S\}$
Range restriction	$r \rhd T$	$\{x \mapsto y \in r \mid y \in T\}$
Range subtraction	$r \ntriangleright T$	$\{x \mapsto y \in r \mid y \notin T\}$

Assume $Prey \in Animal \leftrightarrow Animal$.

We mean hunter $\mapsto$ hunted. The syntax of the relation does not reveal its intended semantics.

- $Mammal \lhd Prey$
- $Mammal \ntriangleleft Prey$
- $Prey \rhd Spiders$
- $Fish \lhd (Prey \rhd Spiders)$
- $Spiders \ntriangleleft (Prey \rhd Spiders)$

Operations on relations

Image	$r[S]$	$\{y \mid x \mapsto y \in r \wedge x \in S\}$
Composition	$p; q$	$\{x \mapsto z \mid x \mapsto y \in p \wedge y \mapsto z \in q\}$
Overriding	$p \Lleftarrow q$	$q \cup (dom(q) \ntriangleleft p)$
Identity	$id(S)$	$\{x \mapsto x \mid x \in S\}$

Overriding:

- Take q, and add the tuples from p whose lhs are not already in q.
- Or, take p and add q, overriding the tuples with the same lhs.

Some useful results, definitions

$$
\begin{aligned}
(r^{-1})^{-1} &= r \\
\mathrm{dom}(r^{-1}) &= \mathrm{ran}(r) \\
(S \lhd r)^{-1} &= r^{-1} \rhd S \\
(p; q)^{-1} &= q^{-1}; p^{-1} \\
p; (q; r) &= (p; q); r \\
p; (q \cup r) &= (p; q) \cup (p; r) \\
(p; q)[S] &= q[p[S]] \\
r[S \cup T] &= r[S] \cup r[T]
\end{aligned}
$$

$r = r^{-1}$	symmetric
$r \cap r^{-1} = \varnothing$	asymmetric
$\mathrm{id}(S) \subseteq r$	reflexive
$r; r \subseteq r$	transitive

Set-theoretic notation more readable than predicate calculus

$$r = r^{-1} \equiv \forall x, y \cdot x \in S \wedge y \in S \Rightarrow (x \mapsto y \in r \Leftrightarrow y \mapsto x \in r)$$

Functions

- Functions: one type of relations.
- Notation: $f(x) = y \equiv x \mapsto y \in f$.
- Every element in domain relates only to one element in range.

$$x \mapsto y \in f \wedge x \mapsto z \in f \Rightarrow y = z$$

- WD conditions:
  - $f \in S \nrightarrow T$
  - $x \in \mathrm{dom}(f)$
- Using right type of function allows different proofs.

Total function ($dom(f) = S$)	$S \rightarrow T$
Partial function	$S \nrightarrow T$

Injection: if $f(x) = f(y)$, then $x = y$.

Partial injection	$S \rightarrowtail\!\!\!\!\!\!+\ T$
Total injection	$S \rightarrowtail T$

Surjection: $f \in S \leftrightarrow T$, $ran(f) = T$.

Partial surjection	$S \twoheadrightarrow\!\!\!\!\!\!+\ T$
Total surjection	$S \twoheadrightarrow T$

Bijection	$S \rightarrowtail\!\!\!\!\!\twoheadrightarrow T$

An example of functions and relations: a strict society

- Every person is either a man or a woman.
- No person is man and woman at the same time.
- Only women have husbands, who must be a man.
- Woman have at most one husband.
- Men have at most one wife.
- Mother are married women.

An example of functions and relations: a strict society

Every person is man or woman	$men \subseteq PERSON$

An example of functions and relations: a strict society

Every person is man or woman	$men \subseteq PERSON$
No person is man and woman	$women = PERSON \setminus men$

An example of functions and relations: a strict society

Every person is man or woman	$\mathit{men} \subseteq \mathit{PERSON}$
No person is man and woman	$\mathit{women} = \mathit{PERSON} \setminus \mathit{men}$
Women have husbands (men)	
At most one husband per woman	$\mathit{husband} \in \mathit{women} ⤔ \mathit{men}$
Men at most one wife	

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Let us derive some relations (Double check with Rodin)

$\mathit{wife} =$

$\mathit{spouse} =$

$\mathit{father} =$

$\mathit{children} =$

$\mathit{daughter} =$

$\mathit{sibling} =$

$\mathit{brother} =$

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$\mathit{father} =$

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An example of functions and relations: a strict society

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No person is man and woman	$women = PERSON \setminus men$
Women have husbands (men)	
At most one husband per woman	$husband \in women ⤔ men$
Men at most one wife	
Mother are married women	$mother \in PERSON ⇸ \mathrm{dom}(husband)$

Let us derive some relations (Double check with Rodin)

$wife = husband^{-1}$
$spouse = husband \cup wife$
$father =$
$children =$

$daughter =$
$sibling =$
$brother =$

An example of functions and relations: a strict society

Every person is man or woman	$men \subseteq PERSON$
No person is man and woman	$women = PERSON \setminus men$
Women have husbands (men)	
At most one husband per woman	$husband \in women ⤔ men$
Men at most one wife	
Mother are married women	$mother \in PERSON ⇸ \mathrm{dom}(husband)$

Let us derive some relations (Double check with Rodin)

$wife = husband^{-1}$
$spouse = husband \cup wife$
$father = mother; husband$
$children =$

$daughter =$
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Properties

$$\begin{aligned}
mother &= father; wife \\
spouse &= spouse^{-1} \\
sibling &= sibling^{-1} \\
cousin &= cousin^{-1} \\
father; father^{-1} &= mother; mother^{-1} \\
father; mother^{-1} &= \varnothing \\
mother; father^{-1} &= \varnothing \\
father; children &= mother; children
\end{aligned}$$

Arithmetic

- The usual (+, -, *, ÷) plus: mod, ^ (power).
- card(set), min(set), max(set)

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Wiley-Blackwell, 2nd edition, 211.
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